GENERALIZED VANDERMONDE DETERMINANTS.

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Contents

Abstract. We present a new expression for the generalized Vandermonde determinant [7], [32], and thus for the Schur function. We also obtain an equivalence relation on the set of all generalized Vandermonde determinants.

Keywords: generalized Vandermonde determinant; Schur function; equivalence relation; skew-symmetric polynomials

1. Introduction.

This paper is organized as follows: In this chapter we give a general background. In chapter 2 we give some important definitions to be used in the following. In chapter 3 we prove an expression for a generalized Vandermonde determinant (from now on abbreviated as GVD) with $\lambda_1 = 2$ in terms of the elementary symmetric polynomials e_n. In chapter 4 we prove an expression for an arbitrary GVD and finally in chapter 5 we discuss how the new results are related to previous work.

The GVD are intimately connected to the symmetric group and to partitions as was outlined in [18].

A partition [18] of $\eta \in \mathbb{N}$ is any finite sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{1}$$

of non-negative integers in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n \geq 0$$

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containing only finitely many non-zero terms such that the weight of λ

$$\sum_{j=1}^{l(\lambda)} \lambda_j = \eta, \tag{2}$$

where $l(\lambda)$, the number of parts > 0 of λ, is called the length of λ. We shall find it convenient not to distinguish between two such sequences which differ only by a string of zeros at the end.

Definition 1. If $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}^n$ is a so-called vector of exponents, we define

$$|\epsilon| = \begin{vmatrix} x_1^{\epsilon_n} & \vdots & x_n^{\epsilon_n} \\ \vdots & \ddots & \vdots \\ x_1^{\epsilon_1} & \vdots & x_n^{\epsilon_1} \end{vmatrix}. \tag{3}$$

There are many definitions of GVD and in the computations we will use the one from Heineman [12, p.465], which goes as follows. This notation is a slight variation of the notation a_ϵ by Macdonald [18].

Definition 2. Given partitions $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ of η and $\delta : (n-1, n-2, \dots, 1, 0)$ of $\binom{n}{2}$, the GVD is defined by

$$|\lambda + \delta|. \tag{4}$$

The Schur function s_λ, defined by

$$s_\lambda = \frac{|\lambda + \delta|}{|\delta|}, \tag{5}$$

is a quotient of two homogeneous skew-symmetric polynomials and is a homogeneous symmetric polynomial [18].

Schur functions are also treated, but from a different viewpoint in [20, p.331].

Schur functions have many applications and we give a brief summary. The Schur functions are particularly relevant to discussions of the quantum Hall effect [28], [25]; to the characters of irreducible representations of $U(n)$ [34, p. 213], [28]; to the characters of $Gl(n, \mathbb{C})$, which can be expressed in terms of Schur functions [24],[1, p. 237], [33, ch. VII.6] ; to the characters of $Sp(2n+1, \mathbb{C})$ [19]; to the characters of $Sp(2n, \mathbb{R})$ [11]; and to the characters of the simple Lie algebras $sl(n, \mathbb{C})$ and $su(n)$, which have the same representations [8]. Recently, Schur functions have been used in q-calculus [15], [23]. Also Vandermonde determinants have found applications in numerical analysis [3], [14]. Further GVD occur in Sato theory, where the variables are partial differential operators [4],[22].

One formula for the Schur function is given by the Frobenius character formula for S_η [26].

Theorem 1.1. *Let the order of the centralizer of any permutation* $\in \mu$ [9] *be given by*

$$c_\mu = \prod_{j=1}^{\eta} \mu_j!\ j^{\mu_j}. \tag{6}$$

Let

$$S_\mu = \prod_{j=1}^{\eta} (\sum_{k=1}^{n} x_k^j)^{\mu_j}. \tag{7}$$

Let S_η^* *denote the set of all conjugacy classes of* S_η*. Then for every* λ*,*

$$s_\lambda = \sum_{\mu \in S_\eta^*} \left(\chi_\lambda(\mu) c_\mu^{-1} S_\mu \right). \tag{8}$$

2. Preliminaries.

In this chapter we give some important definitions. In the whole paper, $\widehat{}$ with a subscript means that this subscript is deleted.

Definition 3. The following notation will be used.

$$\mathbf{k} = (k_1, \ldots, k_N),\ 1 \le k_j \le n, 1 \le j \le N. \tag{9}$$

We make the convention that empty products are interpreted as 1, and empty sums are interpreted as 0. The number of variables is denoted by n, as in (3). However, the n in definition 4 is to be interpreted as the temporary number of variables. One example is (50), where the temporary number of variables is $n + \sigma - 1 - m$, which is clear from the context.

We will use the following notation for a Vandermonde determinant, where the variables with index in $\mathbf{k}$ are missing.

Definition 4.

$$|\widehat{\delta_{\mathbf{k}}}| = \prod_{\substack{1 \le j < i \le n, \\ i,j \ne \{\mathbf{k}\}}} (x_i - x_j). \tag{10}$$

Definition 5. The number of missing exponents in a GVD is $s = \lambda_1$. Let C denote the complement operator in $\{1, \ldots, n+s-2\}$. Then the missing exponents in a GVD are the exponents

$$\mathbf{l} = \mathbf{l}(\lambda) = (l_1, \ldots, l_s) = (\lambda + \delta)^C,\ 0 < l_1 < \ldots < l_s < n + s - 1. \tag{11}$$

Conversely, given a sequence of missing exponents, $\lambda(\mathbf{l})$ is defined as the inverse of $\mathbf{l}(\lambda)$.

To keep track of the missing exponents, we will use both λ and $\mathbf{l}$ to characterize $|\lambda + \delta|$. The $\mathbf{l}$ is useful when $\lambda + \delta$ contains many missing exponents.

Given $\mathbf{l}$ we may define a GVD as in definition 2. We shall use $\lambda(\mathbf{l})$ as above. Denote

$$|\widehat{\epsilon_{l_1,\ldots,l_s}}| = |\lambda(\mathbf{l}) + \delta|. \tag{12}$$

The following fundamental theorem [12, p.466] has been known for many years.

Theorem 2.1. *Let* λ *be the partition of* $n - l_1$ *where*

$$\lambda_1 = \ldots = \lambda_{n-l_1} = 1,\ \lambda_{n-l_1+1} = 0,\ l_1 < n.$$

Then

$$|\widehat{\epsilon_{l_1|}} = |\delta| e_{n-l_1}(x_1, \ldots, x_n). \tag{13}$$

We will now define some numbers which will be used throughout this paper:

Definition 6. Assume that λ and the corresponding GVD are given. Put

$$N = l_{s-1} - s + 2,\ u = \lambda_1 - 2. \tag{14}$$

Let $\{b_{j,N,u}\}_{j=1}^{N}$ be natural numbers, given recursively by

$$b_{j,N,u} = min(\lambda_{n-j} + 1 - \lambda_{n-j+1}, u + 1 - \sum_{k=1}^{j-1} (b_{k,N,u} - 1)), j = 1, \ldots, N-1, \tag{15}$$

$$b_{N,N,u} = min(\lambda_{n-N} + 1 - \lambda_{n-N+1}, u + 2 - \sum_{k=1}^{N-1} (b_{k,N,u} - 1)). \tag{16}$$

The reason for the $u + 2$ in the second formula is that the last factor in theorem 3.4 contains a square.

Then the $b_{j,N,u}$ satisfy

$$b_{N,N,u} - 2 + \sum_{j=1}^{N-1} (b_{j,N,u} - 1) = u. \tag{17}$$

These numbers also satisfy the inequalities

$$b_{j,N,u} \geq 1,\ 1 \leq j \leq N-1;\ b_{N,N,u} \geq 2. \tag{18}$$

The maximum value of $b_{j,N,u}$ is in fact equal to the jumps in degree (for $j = 1, \ldots, N$) of the GVD as the following equation shows:

$$max(b_{j,N,u}) = \lambda_{n-j} + 1 - \lambda_{n-j+1},\ j = 1, \ldots, N. \tag{19}$$

For $u = 0$ we obtain

$$b_{j,N,0} = 1,\ j = 1, \dots, N-1,\ b_{N,N,0} = 2. \tag{20}$$

We will later give some examples of how to compute the $b_{j,N,u}$.

The following numbers will be used to keep track of the sign of permutations. As these numbers only occurs as a plus or minus sign, it is enough to compute them mod 2.

Definition 7. Let $I(\pi)_N$ be the number of inversions mod 2 of the permutation [10] $\pi = (k_1, \dots, k_N)$.

The following notation will be used throughout this paper:

Definition 8.

$$(x_1 \dots \prod_{i=1}^{j} \widehat{x_{k_i}} \dots x_n) = \frac{x_1 \dots x_n}{\prod_{i=1}^{j} x_{k_i}} \tag{21}$$

To simplify notation, we introduce the following function $\Theta_\lambda(e)$.

Definition 9. Let $0 < l_1 < \dots < l_s < n + s - 1$,
and let λ be the corresponding partition.
For convenience put

$$N = l_{s-1} - s + 2. \tag{22}$$

Let $U_{n,N}$ be the subset of $\{1, \dots, n\}^N$, where no repetitions are allowed. Then the function $\Theta_\lambda(e) : \mathbb{C}^n \to \mathbb{C}$ is given by

$$\begin{aligned}\Theta_\lambda(e) = (-1)^{\binom{N+1}{2}} \sum_{\mathbf{k} \in U_{n,N}} \prod_{j=1}^{N} (x_1 \dots \prod_{i=1}^{j} \widehat{x_{k_i}} \dots x_n)^{b_{j,N,s-2}} \times \\ \times e_{n-l_s+s-1}(x_1, \dots, \widehat{x_{k_1}}, \dots, \widehat{x_{k_N}}, \dots, x_n) \times \\ \times (-1)^{k_1 + \dots + k_N + I(\pi)_N} |\widehat{\delta_{\mathbf{k}}}|.\end{aligned} \tag{23}$$

We will later see that the above function is a skew-symmetric polynomial of homogenous degree $\binom{n+s}{2} - \sum_{j=1}^{s} l_j$.

Definition 10. For the special case $s = 2$, if l_1 and l_2 are specified, $\lambda(l)$ is defined. For this λ, let $\theta(n, l_1, l_2)$ be $\Theta_\lambda(e)$ as defined in (23).

We give some examples.

Example 1.

$$\theta(n, 0, l_2) = |\delta| x_1 \dots x_n e_{n-l_2+1}(x_1, \dots, x_n); \tag{24}$$

$$\begin{aligned}\theta(n,1,l_2) = &-\sum_{k_1=1}^{n}(-1)^{k_1}(x_1\ldots\widehat{x_{k_1}}\ldots x_n)^2\times \\ &\times|\widehat{\delta_{\mathbf{k}}}|e_{n-l_2+1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,x_n).\end{aligned} \tag{25}$$

If $\lambda = (3,3,0,0,0)$,

$$\begin{aligned}\Theta_\lambda(e) = &\sum_{k_1=1}^{5}\sum_{k_2=1,k_2\neq k_1}^{5}\sum_{k_3=1,k_3\neq k_1,k_3\neq k_2}^{5}(x_1\ldots\prod_{i=1}^{1}\widehat{x_{k_i}}\ldots x_5) \\ &\times(x_1\ldots\prod_{i=1}^{2}\widehat{x_{k_i}}\ldots x_5)(x_1\ldots\prod_{i=1}^{3}\widehat{x_{k_i}}\ldots x_5)^3 \\ &\times e_2(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_3}},\ldots,x_5)(-1)^{k_1+\ldots+k_3+I(\pi)_3}|\widehat{\delta_{\mathbf{k}}}|.\end{aligned} \tag{26}$$

3. The case $\lambda_1 = 2$.

In this chapter we will prove a general equation for a GVD with $\lambda_1 = 2$ in terms of the elementary symmetric polynomials e_n. Expand the determinant

$$\Xi = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ y & x_1 & \cdots & x_n & z \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & x_n^{n+1} & z^{n+1} \end{vmatrix} \tag{27}$$

with respect to column $n+2$.

Definition 11. The minors $\{\Xi_l\}_{l=0}^{n+1}$ are defined by

$$\Xi = \sum_{l=0}^{n+1} z^l(-1)^{l+n+1}\Xi_l \quad . \tag{28}$$

The y and z in (27) are dummy variables, which are used in the computations. The x_j are the variables that will enter in the GVD.

Starting with (13), the strategy in this chapter will be to express these minors in two different ways to obtain an equation which gives an expression for the GVD in terms of multiple sums of elementary symmetric polynomials. As a first step we obtain

Theorem 3.1. *Let $2 \leq l_2 \leq n+1$. Then*

$$|\widehat{\epsilon_{1,l_2}}| = \theta(n,1,l_2). \tag{29}$$

Proof. Our aim is first to expand Ξ_1 with respect to row 1, and then expand all the minors but the first one with respect to column 1 and use (13).

$$
\begin{aligned}
\Xi_1 &= |\delta|(x_1 \dots x_n)^2 + \\
&+ \sum_{k_1=1}^{n} (-1)^{k_1} \begin{vmatrix} y^2 & x_1^2 & \cdots & \widehat{x_{k_1}^2} & \cdots & x_n^2 \\ y^3 & x_1^3 & \cdots & \widehat{x_{k_1}^3} & \cdots & x_n^3 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_{k_1}^{n+1}} & \cdots & x_n^{n+1} \end{vmatrix} = \\
&= |\delta|(x_1 \dots x_n)^2 + \sum_{k_1=1}^{n} (x_1 \dots \widehat{x_{k_1}} \dots x_n)^2 \times \\
&\times \sum_{l_2=2}^{n+1} (-1)^{l_2+k_1} y^{l_2} \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & \cdots & \widehat{x_{k_1}} & \cdots & x_n \\ x_1^2 & \cdots & \widehat{x_{k_1}^2} & \cdots & x_n^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l_2-2}} & \cdots & \widehat{x_{k_1}^{l_2-2}} & \cdots & \widehat{x_n^{l_2-2}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & \widehat{x_{k_1}^{n-1}} & \cdots & x_n^{n-1} \end{vmatrix} = \\
&\overset{\text{by}(13)}{=} |\delta|(x_1 \dots x_n)^2 + \sum_{l_2=2}^{n+1} \sum_{k_1=1}^{n} y^{l_2} e_{n-l_2+1}(x_1, \dots, \widehat{x_{k_1}}, \dots, x_n) \\
&(-1)^{k_1+l_2} (x_1 \dots \widehat{x_{k_1}} \dots x_n)^2 |\widehat{\delta_{\mathbf{k}}}|.
\end{aligned}
\tag{30}
$$

Now expand Ξ_1 with respect to column 1.

$$
\Xi_1 = |\delta|(x_1 \dots x_n)^2 + \sum_{l_2=2}^{n+1} (-1)^{l_2+1} y^{l_2} |\widehat{\epsilon_{1,l_2}}|. \tag{31}
$$

Finally equate the coefficients of y^{l_2}. □

Theorem 3.2. *Let* $3 \leq l_2 \leq n+1$. *Then*

$$
|\widehat{\epsilon_{2,l_2}}| = \theta(n, 2, l_2). \tag{32}
$$

Proof. Expand Ξ_2 with respect to row 1 and expand the last determinant with respect to column 1. Use (13) and (29).

$$\Xi_2 = |\delta| x_1 \ldots x_n e_{n-1}(x_1, \ldots, x_n) + y \sum_{k_1=1}^{n} (x_1 \ldots \widehat{x_{k_1}} \ldots x_n)^3 |\widehat{\delta_{\mathbf{k}}}| (-1)^{k_1} +$$

$$+ \sum_{l=3}^{n+1} y^l \sum_{k_1=1}^{n} (-1)^{k_1+l} \begin{vmatrix} x_1 & \cdots & \widehat{x_{k_1}} & \cdots & x_n \\ x_1^3 & \cdots & \widehat{x_{k_1}}^3 & \cdots & x_n^3 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^l} & \cdots & \widehat{x_{k_1}^l} & \cdots & \widehat{x_n^l} \\ \vdots & \cdots & \vdots & \vdots & \cdots \\ x_1^{n+1} & \cdots & \widehat{x_{k_1}^{n+1}} & \cdots & x_n^{n+1} \end{vmatrix} =$$

$$|\delta| x_1 \ldots x_n e_{n-1}(x_1, \ldots, x_n) + y \sum_{k_1=1}^{n} (x_1 \ldots \widehat{x_{k_1}} \ldots x_n)^3 |\widehat{\delta_{\mathbf{k}}}| (-1)^{k_1} +$$

$$+ \sum_{l=3}^{n+1} y^l \sum_{k_1=1}^{n} (-1)^{k_1+l} x_1 \cdots \widehat{x_{k_1}} \cdots x_n \times$$

$$\times \begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1^2 & \cdots & \widehat{x_{k_1}}^2 & \cdots & x_n^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{x_1^{l-1}} & \cdots & \widehat{x_{k_1}^{l-1}} & \cdots & \widehat{x_n^{l-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^n & \cdots & \widehat{x_{k_1}^n} & \cdots & x_n^n \end{vmatrix} \overset{\text{by}(29)}{=}$$

$$|\delta| x_1 \ldots x_n \; e_{n-1}(x_1, \ldots, x_n) + y \sum_{k_1=1}^{n} (x_1 \ldots \widehat{x_{k_1}} \ldots x_n)^3 |\widehat{\delta_{\mathbf{k}}}| (-1)^{k_1} +$$

$$+ \sum_{l=3}^{n+1} y^l \sum_{k \in U_{n,2}} (x_1 \ldots \widehat{x_{k_1}} \ldots x_n) e_{n-l+1}(x_1, \ldots, \widehat{x_{k_1}}, \ldots, \widehat{x_{k_2}}, \ldots, x_n) \times$$

$$\times (-1)^{l+k_1+k_2+I(\pi)_2} (x_1 \ldots \widehat{x_{k_1}} \ldots \widehat{x_{k_2}} \ldots x_n)^2 |\widehat{\delta_{\mathbf{k}}}|. \tag{33}$$

The factor $(-1)^{I(\pi)_2+1}$ comes from a renumbering of the summation indices. By expanding Ξ_2 with respect to column 1 we get

$$\Xi_2 = |\delta| x_1 \ldots x_n \; e_{n-1}(x_1, \ldots, x_n) - y |\widehat{\epsilon_{1,2}}| + \sum_{l=3}^{n+1} (-1)^{l+1} y^l |\widehat{\epsilon_{2,l}}|. \tag{34}$$

The theorem now follows by equating the coefficients of y^l. □

Next we need to prove the following lemma by induction.

Lemma 3.3. *Let* $1 \leq l \leq n+1$. *Then*

$$\Xi_l = \sum_{k_1=0}^{l-1} y^{k_1}(-1)^{k_1}\theta(n,k_1,l) + \sum_{k_1=l+1}^{n+1} y^{k_1}(-1)^{k_1+1}\theta(n,l,k_1). \tag{35}$$

Proof. The lemma is true for $l=1$ by (25) and (30) and for $l=2$ by (24), (25) and (34). Assume that the induction hypothesis is true for Ξ_{l-1}.

$$\Xi_{l-1} = \sum_{k_1=0}^{l-2} y^{k_1}(-1)^{k_1}\theta(n,k_1,l-1) + \sum_{k_1=l}^{n+1}(-1)^{k_1+1}y^{k_1}\theta(n,l-1,k_1). \tag{36}$$

On the other hand an expansion of Ξ_{l-1} with respect to column 1 gives

$$\begin{aligned} \Xi_{l-1} &= |\delta| x_1 \ldots x_n \; e_{n-l+2}(x_1,\ldots,x_n) \\ &+ \sum_{k_1=1}^{l-2}(-1)^{k_1}y^{k_1}|\widehat{\epsilon_{k_1,l-1}}| + \sum_{k_1=l}^{n+1}(-1)^{k_1+1}y^{k_1}|\widehat{\epsilon_{l-1,k_1}}|. \end{aligned} \tag{37}$$

Equating the coefficients for y^{k_1} of the two last equations gives first (24) and (29), then for $1 < k_1 < l-1$

$$|\widehat{\epsilon_{k_1,l-1}}| = \theta(n,k_1,l-1), \tag{38}$$

and finally for $l-1 < k_1 < n+2$

$$|\widehat{\epsilon_{l-1,k_1}}| = \theta(n,l-1,k_1). \tag{39}$$

The verification of the induction hypothesis is completed by expanding Ξ_l with respect to row 1.

$$\Xi_l = (x_1 \ldots x_n)|n,n-1,\ldots,l,l-2,\ldots,1,0| +$$

$$+\sum_{k_2=1}^{n}(-1)^{k_2}\begin{vmatrix} y & x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n \\ y^2 & x_1^2 & \cdots & \widehat{x_{k_2}^2} & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{y^l} & \widehat{x_1^l} & \cdots & \widehat{x_{k_2}^l} & \cdots & \widehat{x_n^l} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ y^{n+1} & x_1^{n+1} & \cdots & \widehat{x_{k_2}^{n+1}} & \cdots & x_n^{n+1} \end{vmatrix}.$$

Expand the last determinant with respect to column 1 and use (29), (38) and (39).

$$\Xi_l = \sum_{k_1=0}^{1} y^{k_1}(-1)^{k_1}\theta(n,k_1,l) + \sum_{k_2=1}^{n}\sum_{k_1=2}^{l-1}(-1)^{k_2+k_1+1}y^{k_1}\times$$

$$\times x_1\cdots\widehat{x_{k_2}}\cdots x_n\begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1\\ x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^{k_1-1}} & \cdots & \widehat{x_{k_2}^{k_1-1}} & \cdots & \widehat{x_n^{k_1-1}}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^{l-1}} & \cdots & \widehat{x_{k_2}^{l-1}} & \cdots & \widehat{x_n^{l-1}}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_1^n & \cdots & \widehat{x_{k_2}^n} & \cdots & x_n^n \end{vmatrix}+\sum_{k_2=1}^{n}\sum_{k_1=l+1}^{n+1}$$

$$(-1)^{k_2+k_1}y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n\times\begin{vmatrix} 1 & \cdots & \widehat{1} & \cdots & 1\\ x_1 & \cdots & \widehat{x_{k_2}} & \cdots & x_n\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^{l-1}} & \cdots & \widehat{x_{k_2}^{l-1}} & \cdots & \widehat{x_n^{l-1}}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \widehat{x_1^{k_1-1}} & \cdots & \widehat{x_{k_2}^{k_1-1}} & \cdots & \widehat{x_n^{k_1-1}}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ x_1^n & \cdots & \widehat{x_{k_2}^n} & \cdots & x_n^n \end{vmatrix}=$$

$$=\sum_{k_1=0}^{1}y^{k_1}(-1)^{k_1}\theta(n,k_1,l)+\sum_{k_2=1}^{n}\sum_{k_1=2}^{l-1}(-1)^{k_2+1+k_1}\times$$

$$\times y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n\theta(n-1,k_1-1,l-1)(x_1,\ldots,\widehat{x_{k_2}},\ldots,x_n)+$$

$$+\sum_{k_2=1}^{n}\sum_{k_1=l+1}^{n+1}(-1)^{k_2+k_1}y^{k_1}x_1\cdots\widehat{x_{k_2}}\cdots x_n\times$$

$$\times\theta(n-1,l-1,k_1-1)(x_1,\ldots,\widehat{x_{k_2}},\ldots,x_n)=\sum_{k_1=0}^{l-1}y^{k_1}(-1)^{k_1}\theta(n,k_1,l)+$$

$$+\sum_{k_1=l+1}^{n+1}y^{k_1}(-1)^{k_1+1}\theta(n,l,k_1).$$

□

We immediately obtain a general theorem for a GVD with $\lambda_1=2$.

Theorem 3.4. *Let* $0<l_1<l_2<n+1$. *Then*

$$|\widehat{\epsilon_{l_1,l_2}}|=\theta(n,l_1,l_2). \tag{40}$$

Proof. This follows from (38) or (39). □

4. General GVD.

We are now going to prove an equation for a general GVD. This equation will be a natural generalization of theorem 3.4. The main theorem defines an equivalence relation E on the set of all GVD.

Definition 12. Let n denote the number of variables in the GVD, and let λ be the partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non-negative integers in decreasing order.

The equivalence class E_{n+s,l_{s-1},l_s} is defined by the following three criteria:

(1) The highest power in the determinant is $n+s-1$, a constant.
(2) $l_s(\lambda)$ and $l_{s-1}(\lambda)$ are constant.
(3)
$$\sum_{j=1}^{l_{s-1}-s+2} b_{j,l_{s-1}-s+2,s-2} = l_{s-1} + 1, \tag{41}$$
where $b_{j,l_{s-1}-s+2,s-2}$ is defined by (15)–(16).

Definition 13. An orbit in an equivalence class E_{n+s,l_{s-1},l_s} is a sequence of elements $E(s)$ with increasing values of s, starting with $s = 2$. If, in $E(s)$, the exponents $0 < l_1 < \ldots < l_s < n+s-1$ are missing, $E(s+1)$ is obtained from $E(s)$ by removing an exponent l, where $0 < l < l_1$. Thus, $E(s+1)$ has missing exponents $0 < l_1' < l_2' < \ldots, < l_{s+1}'$, where
$$l_1' = l,\ l_j' = l_{j-1}, j = 2, \ldots, s+1 \tag{42}$$

Definition 14. For every orbit in E_{n+s,l_{s-1},l_s} we define an ordering by $E(s_1) > E(s_2)$ if $s_1 > s_2$.

We can now finally give some examples which show how to compute the $b_{j,t,u}$. In each example we stay in the same orbit.

Example 2. To compute the b:s for the determinant defined by $\lambda = (5,5,0)$ we move in positive direction from the determinant defined by $\lambda = (2,2,0,0,0,0)$, which has $u = 0$, and the b:s are given by (20). Move to $\lambda = (3,3,0,0,0)$, which has $b_{1,3,1} = 1$, $b_{2,3,1} = 1$, $b_{3,3,1} = 3$. Move to $\lambda = (4,4,0,0)$, which has $b_{1,2,2} = 1$, $b_{2,2,2} = 4$. And finally move to $\lambda = (5,5,0)$, which has $b_{1,1,3} = 5$.

Example 3. To compute the b:s for the determinant defined by $\lambda = (4,2,0)$ we move in positive direction from the determinant defined by $\lambda = (2,0,0,0,0)$, which has $u = 0$, and the b:s are given by (20). Move to $\lambda = (3,1,0,0)$, which has $b_{1,3,1} = 1$, $b_{2,3,1} = 2$, $b_{3,3,1} = 2$. And finally move to $\lambda = (4,2,0)$, which has $b_{1,2,2} = 3$, $b_{2,2,2} = 2$.

Example 4. We give an example of three different orbits in $E_{8,4,5}$. We describe each orbit by a sequence of partitions.

Orbit 1:

$$(2,2,0,0,0,0),(3,3,0,0,0),(4,4,0,0),(5,5,0).$$

Orbit 2:

$$(2,2,0,0,0,0),(3,3,1,0,0),(4,4,2,0).$$

Orbit 3:

$$(2,2,0,0,0,0),(3,3,1,1,0).$$

All these orbits can be moved to $(5,5,0)$, the final element of the first orbit, by determinant manipulations, but it is only the first orbit, which follows the procedure in the proof of the main theorem.

The following lemma is necessary for the proof of the main theorem.

Lemma 4.1. *Let $1 \leq m \leq s-2$. We want to compare two permutations: $I(\pi)_{l_{s-1-m}}$, where $\pi = (k_1, \dots, k_{l_{s-1-m}})$ and $I(\pi)_{l_{s-1}+1-m}$, where we have adjoined an extra*

$$k_{J+1} = n + s - 1 - m, \tag{43}$$

subject to the following constraint

$$l_{s-1-m} = \sum_{h=1}^{J} b_{h,l_{s-1}+1-m,m-1}. \tag{44}$$

The $\mathbf{k}$ *are supposed to have the same values in the extented $l_{s-1}+1-m$-tuple except for the reordering induced by (43). The maximum value of k_i is $n+s-2-m$. Then the following equation obtains:*

$$(-1)^{\binom{l_{s-1}+1-m}{2}+I(\pi)_{l_{s-1-m}}} = (-1)^{\binom{l_{s-1}+2-m}{2}+I(\pi)_{l_{s-1}+1-m}+l_{s-1-m}-1}. \tag{45}$$

Proof. We observe that

$$(-1)^{I(\pi)_{l_{s-1-m}}} \equiv \delta + (-1)^{I(\pi)_{l_{s-1}+1-m}} \bmod 2 \tag{46}$$

if

$$J + 1 \equiv \delta + l_{s-1} + 1 - m \bmod 2, \tag{47}$$

where $\delta \in \{0,1\}$, and a simplification shows that we must prove that

$$J \equiv l_{s-1-m} \bmod 2. \tag{48}$$

But the last statement follows from (44). □

We are now ready to prove the main theorem of this paper.

Theorem 4.2. *Let n, σ denote the number of variables and the number of missing exponents in $|\widehat{\epsilon_{l_1,\ldots,l_\sigma}}|$.*

Then

$$|\widehat{\epsilon_{l_1,\ldots,l_\sigma}}| = \Theta_\lambda(e). \tag{49}$$

Proof. We will move within the only orbit in $E_{n+\sigma,l_{\sigma-1},l_\sigma}$, which ends in $|\widehat{\epsilon_{l_1,\ldots,l_\sigma}}|$.

We introduce an 'induction variable' m, which goes from 0 to $\sigma-2$. Every time we move from $E(s)$ to $E(s+1)$, m increases by one, the number of variables in the new determinant decreases by one and the exponent of the extracted monomial decreases.

The induction hypothesis is true for $m=0$ by theorem (3.4).

Assume that the theorem is true for $m-1$.

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+\sigma-1-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_{\sigma-m}}} & \cdots & \widehat{x_{n+\sigma-1-m}^{l_{\sigma-m}}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_\sigma}} & \cdots & \widehat{x_{n+\sigma-1-m}^{l_\sigma}} \\ \vdots & \ddots & \vdots \\ x_1^{n+\sigma-1} & \cdots & x_{n+\sigma-1-m}^{n+\sigma-1} \end{vmatrix} = (-1)^{\binom{l_{\sigma-1}+2-m}{2}} \sum_{\mathbf{k}\in U_{n+\sigma-1-m,l_{\sigma-1}+1-m}} \tag{50}$$
$$\prod_{j=1}^{l_{\sigma-1}+1-m} (x_1 \ldots \prod_{i=1}^{j} \widehat{x_{k_i}} \ldots x_{n+\sigma-1-m})^{b_{j,l_{\sigma-1}+1-m,m-1}} \times$$
$$\times\, e_{n-l_\sigma+\sigma-1}(x_1,,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_{l_{\sigma-1}+1-m}}},\ldots,x_{n+\sigma-1-m})\times$$
$$\times\, (-1)^{k_1+\ldots+k_{l_{\sigma-1}+1-m}+I(\pi)_{l_{\sigma-1}+1-m}}|\widehat{\delta_{\mathbf{k}}}|.$$

An expansion of the same determinant with respect to the last column gives

$$\sum_{l=0}^{l_{\sigma-1-m}} (-1)^{n+\sigma-m+l} x_{n+\sigma-1-m}^{l} \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+\sigma-2-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l}} & \cdots & \widehat{x_{n+\sigma-2-m}^{l}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_\sigma}} & \cdots & \widehat{x_{n+\sigma-2-m}^{l_\sigma}} \\ \vdots & \ddots & \vdots \\ x_1^{n+\sigma-1} & \cdots & x_{n+\sigma-2-m}^{n+\sigma-1} \end{vmatrix} + \\ + G(x_{n+\sigma-1-m}), \tag{51}$$

where $G(x_{n+\sigma-1-m})$ are the terms with $x_{n+\sigma-1-m}$ of order $l_{\sigma-1-m}+1$ and higher.

We now have to pick out the terms which contain $x_{n+\sigma-1-m}^{l_{\sigma-1-m}}$ from (50) i.e. we have to solve the equation

$$l_{\sigma-1-m} = \sum_{h=1}^{J} b_{h,l_{\sigma-1}+1-m,m-1} \tag{52}$$

for J. This implies that

$$k_{J+1} = n+\sigma-1-m. \tag{53}$$

Further the b:s in the new determinant have the following values, which follow from (53).

$$b_{j,l_{\sigma-1}-m,m} = b_{j,l_{\sigma-1}+1-m,m-1},\ 1 \leq j \leq J-1 \tag{54}$$

$$b_{J,l_{\sigma-1}-m,m} = b_{J,l_{\sigma-1}+1-m,m-1} + b_{J+1,l_{\sigma-1}+1-m,m-1} \tag{55}$$

$$b_{j,l_{\sigma-1}-m,m} = b_{j+1,l_{\sigma-1}+1-m,m-1},\ J+1 \leq j \leq l_{\sigma-1}-m \tag{56}$$

In the special case $m=1$,

$$\begin{aligned} &b_{1,l_{\sigma-1}-1,1} = 1, \ldots, b_{l_{\sigma-2},l_{\sigma-1}-1,1} = 2, b_{l_{\sigma-2}+1,l_{\sigma-1}-1,1} = 1, \\ &, \ldots, b_{l_{\sigma-1}-2,l_{\sigma-1}-1,1} = 1, b_{l_{\sigma-1}-1,l_{\sigma-1}-1,1} = 2. \end{aligned} \tag{57}$$

If $l_{\sigma-2}+1 = l_{\sigma-1}$, the last term shall be $b_{l_{\sigma-2},l_{\sigma-1}-1,1} = 3$.

This is accomplished by putting $k_{l_{\sigma-2}+1} = n+\sigma-2$ followed by suppression of the index k_j, which results in a reordering of the k:s.

By equating equations (50) and (51) , cancelling the factor

$(-1)^{n+\sigma-m+l_{\sigma-1-m}}$ and equating the coefficients of $x_{n+\sigma-1-m}^{l_{\sigma-1-m}}$, we obtain

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+\sigma-2-m} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_{\sigma-m-1}}} & \cdots & \widehat{x_{n+\sigma-2-m}^{l_{\sigma-m-1}}} \\ \vdots & \ddots & \vdots \\ \widehat{x_1^{l_\sigma}} & \cdots & \widehat{x_{n+\sigma-2-m}^{l_\sigma}} \\ \vdots & \ddots & \vdots \\ x_1^{n+\sigma-1} & \cdots & x_{n+\sigma-2-m}^{n+\sigma-1} \end{vmatrix} = \tag{58}$$

$$= (-1)^{\binom{l_{\sigma-1}+2-m}{2}+n+\sigma-m+l_{\sigma-1-m}} \sum_{\mathbf{k}\in U_{n+\sigma-2-m,l_{\sigma-1}-m}}$$

$$\prod_{j=1}^{l_{\sigma-1}-m} (x_1 \ldots \prod_{i=1}^{j}\widehat{x_{k_i}} \ldots x_{n+\sigma-2-m})^{b_{j,l_{\sigma-1}-m,m}} \times$$

$$\times\, e_{n-l_\sigma+\sigma-1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_{l_{\sigma-1}-m}}},\ldots,x_{n+\sigma-2-m}) \times$$

$$\times\, (-1)^{k_1+\ldots+k_{l_{\sigma-1}-m}+(n+\sigma-1-m)+I(\pi)_{l_{\sigma-1}+1-m}}|\widehat{\delta_{\mathbf{k}}}|.$$

By lemma 4.1 the last determinant is equal to

$$(-1)^{\binom{l_{\sigma-1}+1-m}{2}} \sum_{\mathbf{k}\in U_{n+\sigma-2-m,l_{\sigma-1}-m}}$$

$$\prod_{j=1}^{l_{\sigma-1}-m} (x_1 \ldots \prod_{i=1}^{j}\widehat{x_{k_i}} \ldots x_{n+\sigma-2-m})^{b_{j,l_{\sigma-1}-m,m}} \times \tag{59}$$

$$\times\, e_{n-l_\sigma+\sigma-1}(x_1,\ldots,\widehat{x_{k_1}},\ldots,\widehat{x_{k_{l_{\sigma-1}-m}}},\ldots,x_{n+\sigma-2-m}) \times$$

$$\times\, (-1)^{k_1+\ldots+k_{l_{\sigma-1}-m}+I(\pi)_{l_{\sigma-1}-m}}|\widehat{\delta_{\mathbf{k}}}|.$$

Now put $m = \sigma - 2$ to obtain equation (49). □

5. Discussion.

Since the subject contains results of many authors, we will add more references to other results and discuss the relations and the differences, between them and the presented author's result, to show its originality and explain its applicability.

Let k be the highest power in $|\lambda + \delta|$. In 1929 E.R. Heineman [12, p.474] gave another expression for a generalized Vandermonde determinant expressed as a quotient of a certain determinant and $|\delta|^{k-n}$.

In 1933 A.Dresden [5] gave another formula, expressing GVD by means of sums of all symmetric polynomials of a given degree. It looks like the present author's result could be better since it is based just on elementary symmetric polynomials.

The $\Theta_\lambda(e)$, where λ runs through all partitions of length $\leq n$, form a basis of the $\mathbb{Z}$-module A_n of skew-symmetric polynomials in $x_1, \ldots, x_n$ [18, p.40]. Just as for the Schur function, λ determines Θ completely and the number of variables equals the number of parts in λ. If we change the e in (23) to either p (power sum) or to h (complete symmetric polynomial), we get other skew-symmetric polynomials and the purpose of another paper could be to study their properties.

The following equivalent definitions of Schur functions are wellknown, see [27], [2].

(1) As the generating function of standard tableaux;
(2) As a ratio of the Vandermonde and a GVD;
(3) Via the Jacobi-Trudi identity

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq n}; \tag{60}$$

(4) Via the Nägelsbach [21] identity

$$s_\lambda = \det\left(e_{\lambda'_i - i + j}\right)_{1 \leq i,j \leq l(\lambda')}. \tag{61}$$

They proved the equivalence of the four definitions using lattice path methods.

The following formula [6] is equivalent to the Nägelsbach-Kostka formula (61).

Theorem 5.1. *Let $\mathbf{k} = (k_0, \ldots, k_{s-1})$ be the set of all permutations of $(0, \ldots, s-1)$ and let λ be a given partition. We tacitly assume that our symmetric polynomials are functions of the variables $x_1, \ldots, x_n$. Then*

$$s_\lambda = \sum_{\mathbf{k}} (-1)^{\binom{s}{2} + I(\pi)_s} \prod_{t=0}^{s-1} e_{n+k_t-l_{s-t}}. \tag{62}$$

This formula also expresses the matrix elements for the transition matrix between s_λ and e_λ. It has been given in some special cases in Kostka [16, p.118-120].

The formula (62) has some similarities with the main theorem (49). The factor $(-1)^{I(\pi)}$ pops up in both of them and this is quite natural, since we work with combinatorics.

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