SOME RESULTS FOR q–FUNCTIONS OF MANY VARIABLES II

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This is a continuation of [19], where we presented an extension of the q-hypergeometric function with connection to the title of this paper. In chapter one we present some quadratic $q-$ hypergeometric transformations, to give more examples of this extension. In chapter two, systems of partial q-difference equations for the q-Appell and q-Lauricella functions are presented in the authors notation. Other attempts to find these equations were made by Jackson. It turns out that these q-difference equations can be written in many equivalent forms, which gives rise to the notion of equivalence class for q-difference equations. In chapter three q-analogues of expansion formulas by Chaundy [11] and Burchnall & Chaundy [9] are found. In the process we obtain a corrected version of [26, (55) p.79]. In chapter four we find an expansion formula for a $\Phi^{2:1}_{2:0}$ function by using Jackson's sum of a terminating very-well-poised balanced ${}_8\phi_7$ series. We also find the corresponding q-binomial identity.

1. Some quadratic $q-$hypergeometric transformations

Kummer [31] first found all the solutions of the hypergeometric differential equation. Then Kummer substituted various fractional transformations

$$x \to \frac{\sqrt{r}+\sqrt{p}}{\sqrt{r}-\sqrt{p}},\ r=(a+bx)^2,\ p=a'+2b'x+c'x^2, \tag{1}$$

to see if this new hypergeometric function satisfied the same differential equation. This led to an impressive list of so-called quadratic transformations. In modern notation, these have a slightly different form. Two examples are [5, p. 125 (3.1.4)],[25, p. 169], [8, p. 9 (2)], [31, p. 78 (53)], [33, p. 67 (3)], [23, p. 59 (3.1.4)].

$${}_2F_1(a,b;1+a-b;x)=(1-x)^{-a}\ {}_2F_1(\tfrac{a}{2},\tfrac{a+1}{2}-b;1+a-b;\frac{-4x}{(1-x)^2}). \tag{2}$$

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[31, p. 78 (51)], [23, p. 68 (3.5.6)].
(3)
$$ {}_2F_1(a,b;1+a-b;x^2)=(1-x)^{-2a}\,{}_2F_1(a,a+\tfrac{1}{2}-b;1+2a-2b;\frac{-4x}{(1-x)^2}). $$

We are going to find q-analogues of these formulas here. Because the proofs involve the q-binomial theorem, the obtained formulas are only formal. It should however be possible to find some analytic (meromorphic) continuation to continue the formulas outside the convergence region in the spirit of [29]. There are some similar approaches to this formal procedure in the literature. In [13] a parameter augmentation method for a reciprocal of a q-shifted factorial was used to obtain q-summation formulas. In [17], [18] the equivalent approach (by the q-binomial theorem) to use the q-exponential function E_q to obtain formulas for q-Laguerre polynomials was used. For a short discussion of generalization to n variables see [18]. Now back to quadratic transformations.

Theorem 1.1. *A q-analogue of (2).*

(4)
$$ {}_2\phi_1(a,b;1+a-b|q,zq^{\frac{a+1}{2}-b})\cong\sum_{m=0}^{\infty}\frac{\langle\frac{a}{2},\frac{a+1}{2}-b,\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}};q\rangle_m(-z)^m q^{-\binom{m}{2}}}{\langle 1,1+a-b;q\rangle_m(zq^{-m};q)_{a+2m}} $$
$$ \equiv\ {}_5\phi_{3,\ k}\left[\begin{matrix}\frac{a}{2},\frac{a+1}{2}-b,\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}},\infty\\ 1+a-b\end{matrix}|q;z|||\begin{matrix}-\\ (zq^{-k};q)_{a+2k}\end{matrix}\right], $$

where the symbol $\cong$ denotes that the equality is purely formal.
A more practical form is the finite version

(5)
$$ {}_2\phi_1(-n,b;1-n-b|q,zq^{\frac{-n+1}{2}-b})=\sum_{m=0}^{\frac{n}{2}}\frac{\langle\frac{-n}{2},\frac{-n+1}{2}-b,\widetilde{\frac{-n}{2}},\widetilde{\frac{-n+1}{2}};q\rangle_m(-z)^m q^{-\binom{m}{2}}}{\langle 1,1-n-b;q\rangle_m(zq^{-m};q)_{-n+2m}} $$
$$ \equiv\ {}_5\phi_{3,\ k}\left[\begin{matrix}\frac{-n}{2},\frac{-n+1}{2}-b,\widetilde{\frac{-n}{2}},\widetilde{\frac{-n+1}{2}},\infty\\ 1-n-b\end{matrix}|q;z|||\begin{matrix}-\\ (zq^{-k};q)_{-n+2k}\end{matrix}\right]. $$

Proof. By the q-binomial theorem, the RHS can be written

$$ (6)\qquad \sum_{m,k=0}^{\infty}\frac{\langle\frac{a}{2},\frac{a+1}{2}-b,\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}};q\rangle_m\langle a+2m;q\rangle_k(-z)^m z^k q^{-mk-\binom{m}{2}}}{\langle 1,1+a-b;q\rangle_m\langle 1;q\rangle_k}. $$

The coefficient for z^n is

(7)
$$\sum_{m=0}^{n}\frac{\langle \frac{a}{2},\frac{a+1}{2}-b,\widetilde{\frac{a}{2}},\widetilde{\frac{a+1}{2}};q\rangle_m\langle a+2m;q\rangle_{n-m}(-1)^m q^{\frac{m^2+m}{2}-nm}}{\langle 1,1+a-b;q\rangle_m\langle 1;q\rangle_{n-m}}=$$
$$\frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n}\sum_{m=0}^{n}\frac{\langle a+n,-n,\frac{a+1}{2}-b;q\rangle_m}{\langle 1+a-b,1,\frac{a+1}{2};q\rangle_m}q^m=\frac{\langle 1-b-n,a,\frac{a+1}{2};q\rangle_n}{\langle 1+a-b,1,\frac{1-a}{2}-n;q\rangle_n}=$$
$$\frac{\langle b,a;q\rangle_n}{\langle 1+a-b,1;q\rangle_n}q^{n(\frac{1+a}{2}-b)}.$$

□

To find the second q-analogue, we will use the following formula which follows from [4].

Lemma 1.2. *An improved version of* [37, p. 426, 2.2]

(8)
$${}_4\phi_3\left[\begin{matrix}a+\frac{1}{2}+l,\widetilde{a+\frac{1}{2}+l},2a+2n,-2n\\ a+\frac{1}{2},\widetilde{a+\frac{1}{2}},2a+1+2l\end{matrix}|q,q\right]=q^{2n(a+\frac{1}{2}+l)}\frac{\langle\frac{1}{2},-l;q^2\rangle_n}{\langle a+\frac{1}{2},1+a+l;q^2\rangle_n}.$$

$$(9)\qquad 4\phi_3\left[\begin{matrix}a+\frac{1}{2}+l,\widetilde{a+\frac{1}{2}+l},2a+n,-n\\ a+\frac{1}{2},\widetilde{a+\frac{1}{2}},2a+1+2l\end{matrix}|q,q\right]=0,\ n\text{ odd}.$$

Theorem 1.3. *A q–analogue of (3).*

(10)
$${}_2\phi_1\left[\begin{matrix}a,-l\\ a+1+l,\end{matrix}|q^2,y^2\right]=$$
$${}_5\phi_{3,\ k}\left[\begin{matrix}a+l+\frac{1}{2},\widetilde{a+l+\frac{1}{2}},a,\widetilde{a},\infty\\ 2a+1+2l\end{matrix}|q;yq^{-a-l-\frac{3}{2}}|||\ \begin{matrix}-\\ (yq^{-k-a-\frac{1}{2}-l};q)_{2a+2k}\end{matrix}\right].$$

Proof.

$$
{}_2\phi_1\left[\begin{matrix} a,-l \\ a+1+l, \end{matrix} |q^2,y^2\right] = \sum_{n=0}^{\infty} \frac{\langle a,-l;q^2\rangle_n}{\langle 1,a+1+l;q^2\rangle_n} y^{2n} =
$$

$$
\sum_{n=0}^{\infty} \frac{\langle 2a;q\rangle_{2n}\langle \frac{1}{2},-l;q^2\rangle_n}{\langle 1;q\rangle_{2n}\langle a+\frac{1}{2},a+1+l;q^2\rangle_n} y^{2n} \overset{\text{by(8)}}{=} \sum_{n=0}^{\infty} \frac{\langle 2a;q\rangle_n y^n}{\langle 1;q\rangle_n}
$$

$$
q^{-n(a+\frac{1}{2}+l)} \sum_{k=0}^{n} \frac{\langle -n,2a+n,a+l+\frac{1}{2},\widetilde{a+l+}\frac{1}{2};q\rangle_k q^k}{\langle a+\frac{1}{2},\widetilde{a+}\frac{1}{2},1,2a+1+2l;q\rangle_k} =
$$

$$
\sum_{k=0}^{\infty} \frac{\langle a+l+\frac{1}{2},\widetilde{a+l+}\frac{1}{2};q\rangle_k\langle a;q^2\rangle_k q^k}{\langle 1,2a+1+2l;q\rangle_k} q^{-k(a+\frac{1}{2}+l)}
$$

$$
\sum_{n=0}^{\infty} \frac{\langle -n-k;q\rangle_k y^{n+k}\langle 2a+2k;q\rangle_n}{\langle 1;q\rangle_{n+k}} q^{-n(a+\frac{1}{2}+l)} = \tag{11}
$$

$$
\sum_{k=0}^{\infty} \frac{\langle a+l+\frac{1}{2},\widetilde{a+l+}\frac{1}{2};q\rangle_k}{\langle 1,2a+1+2l;q\rangle_k} \langle a;q^2\rangle_k q^{-k(a+l)-\frac{k^2}{2}}(-y)^k
$$

$$
\sum_{n=0}^{\infty} \frac{y^n\langle 2a+2k;q\rangle_n}{\langle 1;q\rangle_n} q^{-n(k+a+\frac{1}{2}+l)} = \sum_{k=0}^{\infty} \frac{\langle a+l+\frac{1}{2},\widetilde{a+l+}\frac{1}{2};q\rangle_k}{\langle 1,2a+1+2l;q\rangle_k}
$$

$$
\langle a;q^2\rangle_k \frac{q^{-k(a+l+1)-\frac{k^2}{2}}(-y)^k}{(yq^{-k-a-\frac{1}{2}-l};q)_{2a+2k}} = RHS.
$$

□

For another q–analogue see [23, p. 68].

2. Systems of partial q-difference equations for the q-Appell and q-Lauricella functions

In 1880 Paul Emile Appell (1855-1930) [6], [7] introduced some 2-variable hypergeometric series now called Appell functions.

$$
F_1(a;b,b';c;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b)_{m_1}(b')_{m_2}}{m_1!m_2!(c)_{m_1+m_2}} x_1^{m_1}x_2^{m_2}, \tag{12}
$$

$$
\max(|x_1|,|x_2|) < 1.
$$

$$
F_2(a;b,b';c,c';x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b)_{m_1}(b')_{m_2}}{m_1!m_2!(c)_{m_1}(c')_{m_2}} x_1^{m_1}x_2^{m_2}, \tag{13}
$$

$$
|x_1| + |x_2| < 1.
$$

(14) $$F_3(a,a';b,b';c;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1}(a')_{m_2}(b)_{m_1}(b')_{m_2}}{m_1!m_2!(c)_{m_1+m_2}} x_1^{m_1} x_2^{m_2},$$
$$\max(|x_1|,|x_2|) < 1.$$

(15) $$F_4(a;b;c,c';x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b)_{m_1+m_2}}{m_1!m_2!(c)_{m_1}(c')_{m_2}} x_1^{m_1} x_2^{m_2},$$
$$|\sqrt{x_1}| + |\sqrt{x_2}| < 1.$$

They have the following q-analogues [26], [27], [3]. The convergence area in the x_1 x_2 plane is slightly larger than for the corresponding Appell functions.

(16)
$$\Phi_1(a;b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}.$$

(17)
$$\Phi_2(a;b,b';c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1} x_2^{m_2}.$$

(18)
$$\Phi_3(a,a';b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}.$$

(19)
$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1} x_2^{m_2}.$$

Definition 1. Partial q-derivatives are denoted $D^2_{q,i,j}$ etc. Let $\{\theta_i\}_q \equiv x_i D_{q,i}$. The following inverse pair of symbolic operators defined in [19], [26] will be used in some of the computations.

(20)
$$\nabla_q(h) \equiv \Gamma_q \left[\begin{array}{c} h, h+\{\theta_1\}_q+\{\theta_2\}_q \\ h+\{\theta_1\}_q, h+\{\theta_2\}_q \end{array} \right], \ \triangle_q(h) \equiv \Gamma_q \left[\begin{array}{c} h+\{\theta_1\}_q, h+\{\theta_2\}_q \\ h+\{\theta_1\}_q+\{\theta_2\}_q, h \end{array} \right].$$

In this chapter we are going to find q-difference equations for q-Appell and q-Lauricella functions. So as a preliminary lemma we need the q-difference equations for a ${}_2\phi_1$ q-hypergeometric series.

Lemma 2.1. *The series ${}_2\phi_1(a,b;c|q,x)$ satisfies the q-difference equation due to Heine*

(21)
$$x(q^c - xq^{a+b+1})D_q^2 + \left[\{c\}_q - (\{a\}_q q^b + \{b\}_q q^a + q^{a+b})x\right] D_q - \{a\}_q\{b\}_q I = 0.$$

Proof. The q-difference equation can be written

$$-x\{\theta+a\}_q\{\theta+b\}_q+\{\theta\}_q\{\theta+c-1\}_q=0. \tag{22}$$

This can be restated as

(23)
$$\begin{aligned}&-x(q^a\{\theta\}_q+\{a\}_q)(q^b\{\theta\}_q+\{b\}_q)+\{\theta\}_q(q^c\{\theta-1\}_q+\{c\}_q)=\\&-xq^{a+b}(qx^2D_q^2+xD_q)-x^2D_q(\{a\}_qq^b+\{b\}_qq^a)-x\{a\}_q\{b\}_q+\\&\{\theta\}_q(q^{c-1}\{\theta\}_q-q^{c-1}+\{c\}_q)=\\&-x^3q^{a+b+1}D_q^2-x^2q^{a+b}D_q-x^2D_q(\{a\}_qq^b+\{b\}_qq^a)-x\{a\}_q\{b\}_q+\\&q^cx^2D_q^2+xD_q\{c\}_q=0.\end{aligned}$$

□

There is also a third form [34, p. 11], which is presented for the generalized series ${}_p\phi_{p-1}(a_1,\ldots,a_p;b_1,\ldots,b_{p-1}|q,z)$. We have put $b_p=1$, and e_k=elementary symmetric polynomial.

$$\sum_{k=0}^{p}(-1)^k(e_k(q^{b_i})q^{-k}-e_k(q^{a_i})x)f(q^kx)=0. \tag{24}$$

This equation can't be transformed to a difference equation, because the coefficients of $f(q^kx)$ are not independent of q.

Some of the following equations have appeared in different form and different notation in [26, p. 79-80]. The partial q-difference equations for the q-Appell functions

$$\Phi_1(a;b,b';c|q;x_1,x_2),\Phi_2(a;b,b';c,c'|q;x_1,x_2),$$

$$\Phi_3(a,a';b,b';c|q;x_1,x_2),\Phi_4(a;b;c,c'|q;x_1,x_2)$$

are in corrected form

(25)
$$\begin{aligned}&x_1(q^c-x_1q^{a+b+1})\epsilon_2D_{q,1,1}^2+x_2\left[q^c+x_1(q^a-q^{a+b}-q^{a+b+1})\right]D_{q,1,2}^2-\{b\}_qq^ax_2D_{q,2}+\\&+\left[\{c\}_q-(\{a\}_qq^b+\{b\}_qq^a+q^{a+b})x_1\right]D_{q,1}-\{a\}_q\{b\}_qI=0.\end{aligned}$$

(26)
$$\begin{aligned}&x_1(q^c-x_1q^{a+b+1}\epsilon_2)D_{q,1,1}^2+x_1x_2(q^a-q^{a+b}-q^{a+b+1})D_{q,1,2}^2-\{b\}_qq^ax_2D_{q,2}+\\&+\left[\{c\}_q-(\{a\}_qq^b+\{b\}_qq^a+q^{a+b})x_1\right]D_{q,1}-\{a\}_q\{b\}_qI=0.\end{aligned}$$

$$\begin{aligned}&x_1(q^c\epsilon_2-x_1q^{a+b+1})D_{q,1,1}^2+x_2q^cD_{q,1,2}^2+\\&+\left[\{c\}_q-(\{a\}_qq^b+\{b\}_qq^a+q^{a+b})x_1\right]D_{q,1}-\{a\}_q\{b\}_qI=0.\end{aligned} \tag{27}$$

(28)

$$x_1(q^c - x_1 q^{a+b+1}\epsilon_2^2)D^2_{q,1,1} - 2q^{a+b}\epsilon_2 x_1 x_2 D^2_{q,1,2}$$
$$-[\{a\}_q q^b + \{b\}_q q^a + q^{a+b-1}\epsilon_2]x_2 D_{q,2}+$$
$$\left[\{c\}_q - \epsilon_2(\{a\}_q q^b + \{b\}_q q^a + \epsilon_2 q^{a+b})x_1\right] D_{q,1} - q^{a+b}x_2^2 D^2_{q,2,2} - \{a\}_q\{b\}_q I = 0.$$

The proof of (25) goes as follows: Write the first q-Appell- function in the form

(29)

$$\Phi_1(a;b,b';c|q;x_1,x_2) = \sum_{m_2=0}^{\infty}\frac{\langle a,b';q\rangle_{m_2}}{\langle 1,c;q\rangle_{m_2}}\sum_{m_1=0}^{\infty}\frac{\langle a+m_2,b;q\rangle_{m_1}}{\langle 1,c+m_2;q\rangle_{m_1}}x_1^{m_1}x_2^{m_2}$$

Then the q-difference equation for the inner sum obtains:

$$\begin{aligned}&\sum_{m_2=0}^{\infty}\frac{\langle a,b';q\rangle_{m_2}}{\langle 1,c;q\rangle_{m_2}}\left[x_1(q^{c+m_2} - x_1 q^{a+b+1+m_2})D^2_{q,1,1}+\right.\\ &+\left[\{c+m_2\}_q - (\{a+m_2\}_q q^b + \{b\}_q q^{a+m_2} + q^{a+b+m_2})x_1\right] D_{q,1}\\ &-\{a+m_2\}_q\{b\}_q I]\sum_{m_1=0}^{\infty}\frac{\langle a+m_2,b;q\rangle_{m_1}}{\langle 1,c+m_2;q\rangle_{m_1}}x_1^{m_1}x_2^{m_2} = 0.\end{aligned} \tag{30}$$

We have

$$\{c+m_2\}_q = \begin{cases}\{c\}_q + q^c\{m_2\}_q\\ \{m_2\}_q + q^{m_2}\{c\}_q.\end{cases} \tag{31}$$

Therefore we get the terms

$$\begin{cases}\{c\}_q D_{q,1} + q^c x_2 D^2_{q,1,2}\\ x_2 D^2_{q,1,2} + \epsilon_2\{c\}_q D_{q,1}\end{cases} \tag{32}$$

In the same way we have

$$-(\{a+m_2\}_q q^b + \{b\}_q q^{a+m_2} + q^{a+b+m_2}) =$$

$$\begin{cases}-(q^b\{a\}_q + q^a\{b\}_q + q^{a+b} + \{m_2\}_q(q^{a+b+1} + q^{a+b} - q^a))\\ -(\{m_2\}_q q^{a+b+1} + q^b\{a+1\}_q + q^{m_2}\{b\}_q).\end{cases} \tag{33}$$

Therefore we get the terms

$$-(\{a\}_q q^b + \{b\}_q q^a + q^{a+b})x_1 D_{q,1} - x_1 x_2(-q^a + q^{a+b} + q^{a+b+1})D^2_{q,1,2} \tag{34}$$

or

$$-(x_1 x_2 q^{a+b+1} D^2_{q,1,2} - [\{a+1\}_q q^b + \epsilon_2\{b\}_q]D_{q,1}). \tag{35}$$

This gives us eight equivalent q-difference equations for Φ_1, 4 equivalent q-difference equations for Φ_2, 2 equivalent q-difference equations for Φ_3 and 16 equivalent q-difference equations for Φ_4. These equations are stated in a different form in [24, p. 299]. The q-difference equation for Φ_1 can be written in the following canonical form, a q-analogue of [32, p. 146].

$$-x_1\{\theta_1+b\}_q\{\theta_1+\theta_2+a\}_q+\{\theta_1\}_q\{\theta_1+\theta_2+c-1\}_q=0. \tag{36}$$

The q-difference equation for Φ_2 can be written in the canonical form

$$-x_1\{\theta_1+a\}_q\{\theta_1+\theta_2+b\}_q+\{\theta_1\}_q\{\theta_1+c-1\}_q=0. \tag{37}$$

The q-difference equation for Φ_3 can be written in the canonical form

$$-x_1\{\theta_1+a\}_q\{\theta_1+b\}_q+\{\theta_1\}_q\{\theta_1+\theta_2+c-1\}_q=0. \tag{38}$$

The q-difference equation for Φ_4 can be written in the canonical form

$$-x_1\{\theta_1+\theta_2+a\}_q\{\theta_1+\theta_2+\theta_2+b\}_q+\{\theta_1\}_q\{\theta_1+c-1\}_q=0. \tag{39}$$

The q-difference equation for Φ_1 can be rewritten in the operator form

$$\begin{aligned}&(q^c-x_1q^{a+b+1})\frac{\epsilon_2}{(1-q)^2}q^{-1}(\epsilon_1^2-(1+q)\epsilon_1+q)+\\&[q^c+x_1(q^a-q^{a+b}-q^{a+b+1})]\frac{1}{(1-q)^2}[1-\epsilon_1][1-\epsilon_2]\\&-\{b\}_q\frac{x_1q^a}{1-q}[1-\epsilon_2]-x_1\{a\}_q\{b\}_q\\&+[\{c\}_q-(\{a\}_qq^b+\{b\}_qq^a+q^{a+b})x_1]\frac{1}{1-q}[1-\epsilon_1]=0\end{aligned} \tag{40}$$

Another q-difference equation satisfied by Φ_1 is (special thanks to Axel Riese for finding this equation using *Mathematica*)

$$x_2\{b'\}_qx_1D_{q,x_1}f-x_1\{b\}_qx_2D_{q,x_2}f+(-x_1q^b+x_2q^{b'})x_2D_{q,x_2}x_1D_{q,x_1}f=0 \tag{41}$$

Theorem 2.2. *Equation (26) is also satisfied by (compare* [34, p. 34 (65)]*, where all the solutions of a homogeneous second order q-difference equation were found).*

$$x_1^{1-c}\Phi_2(a-c+1;b-c+1,b';2-c,c'|q;x_1,x_2).$$

Assume a solution to (26) of the form

$$\sum_{m_1,m_2=0}^{\infty}a_{m_1,m_2}x_1^{m_1+\mu_1}x_2^{m_2+\mu_2}.$$

Then the method of Frobenius gives the indicial equation for the term

$$a_{0,0}x_1^{\mu_1-1}x_2^{\mu_2}.$$

$$\{\mu_1\}_q(\{c\}_q+q^c\{\mu_1-1\}_q)=\{\mu_1\}_q\{\mu_1+c-1\}_q. \tag{42}$$

The partial q-difference equations for the q-Lauricella functions, compare [28, p. 15].

$$\Phi_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n|q;x_1,\ldots,x_n) = \\ = \sum_{\mathbf{m}} \frac{\langle a;q\rangle_{m_1+\ldots+m_n}\langle b_1;q\rangle_{m_1}\ldots\langle b_n;q\rangle_{m_n}\prod_{j=1}^n x_j^{m_j}}{\langle c_1;q\rangle_{m_1}\ldots\langle c_n;q\rangle_{m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}, \tag{43}$$

$$\Phi_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n) = \\ = \sum_{\mathbf{m}} \frac{\prod_{j=1}^n\langle a_j,b_j;q\rangle_{m_j}x_j^{m_j}}{\langle c;q\rangle_{m_1+\ldots+m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}, \tag{44}$$

$$\Phi_C^{(n)}(a,b;c_1,\ldots,c_n|q;x_1,\ldots,x_n) = \\ = \sum_{\mathbf{m}} \frac{\langle a,b;q\rangle_{m_1+\ldots+m_n}\prod_{j=1}^n x_j^{m_j}}{\langle c_1;q\rangle_{m_1}\ldots\langle c_n;q\rangle_{m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}, \tag{45}$$

$$\Phi_D^{(n)}(a,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n) = \\ = \sum_{\mathbf{m}} \frac{\langle a;q\rangle_{m_1+\ldots+m_n}\langle b_1;q\rangle_{m_1}\ldots\langle b_n;q\rangle_{m_n}\prod_{j=1}^n x_j^{m_j}}{\langle c;q\rangle_{m_1+\ldots+m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}}. \tag{46}$$

are (we consider only three variables)

(47)
$$x_1(q^{c_1}-x_1q^{a+b_1+1}\epsilon_2\epsilon_3)D_{q,1,1}^2 - x_2x_1q^{a+b_1}D_{q,1,2}^2 - \\ - \{b_1\}_q q^a x_2 D_{q,2} - \epsilon_2 q^{a+b_1}\theta_1\theta_3 - \{b_1\}_q q^a\epsilon_2\theta_3 + \\ + \left[\{c_1\}_q - (\{a\}_q q^{b_1} + \epsilon_2\epsilon_3(\{b_1\}_q q^a + q^{a+b_1}))x_1\right]D_{q,1} - \{a\}_q\{b_1\}_q I = 0.$$

(48)
$$x_1(q^c\epsilon_2\epsilon_3 - x_1q^{a_1+b_1+1})D_{q,1,1}^2 + x_2q^cD_{q,1,2}^2 \\ + \left[\{c\}_q + q^c\theta_3\epsilon_2 - (\{a_1\}_q q^{b_1} + \{b_1\}_q q^{a_1} + q^{a_1+b_1})x_1\right]D_{q,1} - \{a_1\}_q\{b_1\}_q I = 0.$$

(49)
$$x_1(q^{c_1}-x_1q^{a+b+1}\epsilon_2^2\epsilon_3^2)D_{q,1,1}^2 - 2q^{a+b}\epsilon_2\epsilon_3\theta_1\theta_2 - \\ - q^{a+b}\theta_2^2 - 2q^{a+b}\epsilon_2^2\epsilon_3\theta_1\theta_3 - (\{a\}_q q^b + \{b\}_q q^a)\epsilon_2\theta_3 \\ - (\{a\}_q q^b + \{b\}_q q^a)x_2D_{q,2} - q^{a+b}\epsilon_2^2\theta_3^2 - 2q^{a+b}\epsilon_2\theta_2\theta_3 + \\ + \left[\{c_1\}_q - \epsilon_2\epsilon_3(\{a\}_q q^b + \{b\}_q q^a + \epsilon_2\epsilon_3 q^{a+b})x_1\right]D_{q,1} - \{a\}_q\{b\}_q I = 0,$$

symmetric in a,b.

(50)

$$x_1(q^c - x_1q^{a+b_1+1})\epsilon_2\epsilon_3 D^2_{q,1,1} + x_2\left(q^c - x_1q^{a+b_1}\right)D^2_{q,1,2} -$$
$$\{b_1\}_q q^a x_2 D_{q,2} - q^{a+b_1}\epsilon_2\theta_1\theta_3 - \{b_1\}_q q^a\epsilon_2\theta_3 +$$
$$+\left[\{c\}_q + q^c\epsilon_2\theta_3 - \{a\}_q q^{b_1}x_1 - \epsilon_2\epsilon_3(\{b_1\}_q q^a + q^{a+b_1})x_1\right]D_{q,1} - \{a\}_q\{b_1\}_q I = 0.$$

The two first equations can be generalized to

(51)

$$x_1(q^{c_1} - x_1q^{a+b_1+1}\epsilon_2\epsilon_3)D^2_{q,1,1} - \sum_{l=2}^{n} q^{a+b_1}\prod_{m=2}^{l-1}\epsilon_m\theta_1\theta_l - \sum_{l=2}^{n}\{b_1\}_q q^a\prod_{m=2}^{l-1}\epsilon_m\theta_l$$
$$+\left[\{c_1\}_q - (\{a\}_q q^{b_1} + \epsilon_2\epsilon_3(\{b_1\}_q q^a + q^{a+b_1}))x_1\right]D_{q,1} - \{a\}_q\{b_1\}_q I = 0.$$

(52)

$$x_1(q^c\epsilon_2\epsilon_3 - x_1q^{a_1+b_1+1})D^2_{q,1,1} + \sum_{l=2}^{n} q^c\prod_{m=2}^{l-1}\epsilon_m\theta_l D_{q,1}$$
$$+\left[\{c\}_q - (\{a_1\}_q q^{b_1} + \{b_1\}_q q^{a_1} + q^{a_1+b_1})x_1\right]D_{q,1} - \{a_1\}_q\{b_1\}_q I = 0.$$

The last equation can be generalized to

(53)

$$x_1(q^c - x_1q^{a+b_1+1})\prod_{m=2}^{n}\epsilon_m D^2_{q,1,1} + \sum_{l=2}^{n} q^c\prod_{m=2}^{l-1}\epsilon_m\theta_l D_{q,1}$$
$$-\sum_{l=2}^{n} q^{a+b_1}\prod_{m=2}^{l-1}\epsilon_m\theta_1\theta_l - \sum_{l=2}^{n}\{b_1\}_q q^a\prod_{m=2}^{l-1}\epsilon_m\theta_l +$$
$$\left[\{c\}_q - \{a\}_q q^{b_1}x_1 - \prod_{m=2}^{n}\epsilon_m(\{b_1\}_q q^a + q^{a+b_1})x_1\right]D_{q,1} - \{a\}_q\{b_1\}_q I = 0.$$

3. q-analogues of formulas by Chaundy and Burchnall.

Burchnall & Chaundy [9] gave a number of expansion formulas for hypergeometric series in series of hypergeometric series by using the inverse operators in (20) for $q = 1$. The goal was to throw light on the double hypergeometric functions by expressing them in terms of the more elementary hypergeometric functions. Verma [36] extended the above expansions to Kampé de Fériet functions. The late Professor Forsyth had suggested to Jackson [27] that if the base q in the q-difference operator is replaced by $1+\epsilon$, q-analysis could be used to deal with physical problems in which reality is never in exact accord with physical equations. And indeed, in 1942 [26] and 1944 [27] Jackson published two papers on Dqs.

Then q-analogues of the expansions in [9] were presented, in the order of B-C. In the second paper the confluent cases of these expansions together with two lemmas which follow from the q-Vandermonde summation formula were presented. In 1943 Chaundy [12] presented some similar expansions of two types. R.P. Agarwal visited Bailey at Bedford college in London in the early fifties, and published two papers about q-calculus during this time. In the first paper [1], first some of Jacksons expansions were presented in corrected form. Then q-analogues of Chaundy's expansions of the first type were found. The expansions of the first type use Vandermondes lemma in the proofs, wheras the expansions of the second type use lemma (57).

The following expansion formulas are all special cases of [19, Theorem 5.1, p. 229].

Theorem 3.1. *q-analogue of* [9, (30)]*. The first version of this equation occurred in* [26, (37)p. 75]*. The same corrected version also occurred in* [1, 6.8 p. 193].

$$\begin{aligned}\Phi_1(a;b,b';c|q;x,y) = \sum_{r=0}^{\infty} \frac{\langle c-a,a,b,b';q\rangle_r}{\langle 1,c+r-1;q\rangle_r \langle c;q\rangle_{2r}} x^r y^r q^{ra+r(r-1)} \times \\ {}_2\phi_1(a+r,b+r;c+2r|q,x)\ {}_2\phi_1(a+r,b'+r;c+2r|q,y).\end{aligned} \tag{54}$$

Theorem 3.2. *q-analogue of* [9, (31)].

(55)

$${}_2\phi_1(a,b;c|q,x)\ {}_2\phi_1(a,b';c|q,y) = \sum_{r=0}^{\infty} \frac{(-1)^r \langle a,b,b',c-a;q\rangle_r}{\langle 1,c;q\rangle_r \langle c;q\rangle_{2r}} q^{ra+\binom{r}{2}} x^r y^r \times$$
$$\Phi_1(a+r;b+r,b'+r;c+2r|q;x,y).$$

Theorem 3.3. *q-analogue of* [9, (38)]*. The first version of this equation, with Jackson q-addition, occurred in* [26, (45)p. 76].

$$\begin{aligned}{}_2\phi_1(a,b;c|q,x \oplus_q y) = \sum_{r=0}^{\infty} \frac{\langle b;q\rangle_r \langle a;q\rangle_{2r}}{\langle 1;q\rangle_r \langle c;q\rangle_{2r}} q^{rb+r(r-1)} x^r y^r \times \\ \Phi_1(a+2r;b+r,b+r;c+2r|q;x,y).\end{aligned} \tag{56}$$

We are now going to prove an expansion formula in the spirit of Chaundy. For the proof we need a q-analogue of Chaundy [11, p. 164].

Lemma 3.4. [22]

$$\sum_{r=0}^{R}\sum_{s=0}^{S}\frac{(-1)^{r+s}\langle 1;q\rangle_R\langle 1;q\rangle_S}{\langle 1;q\rangle_r\langle 1;q\rangle_{R-r}\langle 1;q\rangle_s\langle 1;q\rangle_{S-s}}\times$$
$$\frac{\langle c-1;q\rangle_{r+s}\langle c;q\rangle_{2r+2s}}{\langle c-1;q\rangle_{2r+2s}\langle c;q\rangle_{R+S+r+s}}\mathrm{QE}(\binom{r}{2}+\binom{s}{2}+Sr)= \tag{57}$$
$$\begin{cases}1, & \textit{if } R=S=0;\\ 0, & \textit{otherwise.}\end{cases}$$

Theorem 3.5.

$$\Phi_3(A,A';B,B';C|q;x,y)=\sum_{r,s=0}^{\infty}(-1)^{r+s}q^{\binom{s}{2}+\binom{r}{2}}$$
$$\times\frac{\langle a,b;q\rangle_r\langle a',b';q\rangle_s\mathrm{QE}(sr-ms-nr+mn)}{\langle 1;q\rangle_r\langle 1;q\rangle_s\langle c+r+s-1;q\rangle_{r+s}} \tag{58}$$
$$\Phi_{1:2}^{1:3}\left[\begin{matrix}c+r+s-1:A,B,-r;A',B',-s\\ C:a,b;a',b'\end{matrix}|q;q,q\right]x^ry^s$$
$$\Phi_3(a+r,a'+s;b+r,b'+s;c+2r+2s|q;x,yq^{r-m}).$$

Proof. The coefficient for

$$\frac{\langle A,B;q\rangle_m\langle A',B';q\rangle_n}{\langle C;q\rangle_{m+n}\langle 1,a,b;q\rangle_m\langle 1,a',b';q\rangle_n}$$

on the RHS is granted absolute convergence

(59)

$$\sum_{r=m}^{\infty}\sum_{s=n}^{\infty}(-1)^{r+s}\mathrm{QE}(\binom{r}{2}+\binom{s}{2})\mathrm{QE}(sr-ms-nr+mn+m+n)\times$$

$$\frac{\langle c+r+s-1;q\rangle_{m+n}\langle a,b;q\rangle_r\langle a',b';q\rangle_s}{\langle 1;q\rangle_r\langle 1;q\rangle_s\langle c+r+s-1;q\rangle_{r+s}}\langle -r;q\rangle_m\times$$

$$\langle -s;q\rangle_n x^r y^s\Phi_3(a+r,a'+s;b+r,b'+s;c+2r+2s|q;x,yq^{r-m})=$$

$$\sum_{R_1,R_2=0}^{\infty}\sum_{r=m}^{R_1}\sum_{s=n}^{R_2}\frac{\langle a,b;q\rangle_{R_1}\langle a',b';q\rangle_{R_2}}{\langle c+r+s-1;q\rangle_{r+s}}x^{R_1}y^{R_2}\frac{\langle c+r+s-1;q\rangle_{m+n}}{\langle c+2r+2s;q\rangle_{R_1+R_2-r-s}}\times$$

$$\frac{(-1)^{r+s+m+n}\mathrm{QE}(\binom{r}{2}+\binom{s}{2}+\binom{m}{2}+\binom{n}{2}-mr-ns+m+n+R_2r-mR_2-nr+mn)}{\langle 1;q\rangle_{r-m}\langle 1;q\rangle_{s-n}\langle 1;q\rangle_{R_1-r}\langle 1;q\rangle_{R_2-s}}$$

$$=\sum_{R_1,R_2=0}^{\infty}\sum_{r'=0}^{R_1-m}\sum_{s'=0}^{R_2-n}\langle a,b;q\rangle_{R_1}\langle a',b';q\rangle_{R_2}\langle c+r'+m+s'+n-1;q\rangle_{m+n}x^{R_1}y^{R_2}$$

$$\frac{(-1)^{r'+s'}\mathrm{QE}(\binom{r'}{2}+\binom{s'}{2}+r'(R_2-n))}{\langle 1;q\rangle_{r'}\langle 1;q\rangle_{s'}\langle 1;q\rangle_{R_1-r'-m}\langle 1;q\rangle_{R_2-s'-n}\langle c+2r'+2m+2s'+2n;q\rangle_{R_1+R_2-r'-s'-m-n}}$$

$$\times\frac{1}{\langle c+r'+m+s'+n-1;q\rangle_{r'+s'+m+n}}\overset{\text{by}(57)}{=}\langle a,b;q\rangle_m\langle a',b';q\rangle_n x^m y^n.$$

□

We now present some q-analogues of hypergeometric equations in Burchnall-Chaundy II. The first formula is a q-analogue of [10, p. 114 (15)].

Theorem 3.6. [15]

$$\Phi_1(a;b,b';c|q;x,xq^{-b'})={}_2\phi_1(a,b+b';c|q,xq^{-b'}). \tag{60}$$

By (54) we obtain a corrected version of [26, (55) p.79]. *This is a q-analogue of an addition formula in the arguments b,b', and a q-analogue of* [10, p. 114 (16)].

(61)

$${}_2\phi_1(a,b+b';c|q,xq^{-b'})=\sum_{r=0}^{\infty}\frac{\langle c-a,a,b,b';q\rangle_r}{\langle 1,c+r-1;q\rangle_r\langle c;q\rangle_{2r}}x^{2r}q^{-rb'+ra+r(r-1)}$$

$$\times{}_2\phi_1(a+r,b+r;c+2r|q,x){}_2\phi_1(a+r,b'+r;c+2r|q,xq^{-b'}).$$

Theorem 3.7. [26, (56) p.79], *a q-analogue of* [10, p. 114 (17)]. *By (55)*

(62)
$$
{}_2\phi_1(a,b;c|q,x)\,{}_2\phi_1(a,b';c|q,xq^{-b'}) = \sum_{r=0}^{\infty} \frac{(-1)^r \langle a,b,b',c-a;q\rangle_r}{\langle 1,c;q\rangle_r \langle c;q\rangle_{2r}} x^{2r} \times q^{-rb'+ra+\binom{r}{2}}\, {}_2\phi_1(a+r,b+b'+2r;c+2r|q,xq^{-b'}).
$$

In this equation we put

(1) $c = b + b'$
(2) $b = \frac{c+1}{2},\ b' = \frac{c-1}{2}$
(3) $b = b' = \frac{c}{2}$.

The result is the following three equations. Compare [26, (57)-(58) p. 79].

Theorem 3.8.

(63)
$$
\begin{aligned}
&{}_2\phi_1(a,b;b+b'|q,x)\,{}_2\phi_1(a,b';b+b'|q,xq^{-b'}) \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r \langle a,b,b',b+b'-a;q\rangle_r}{\langle 1,b+b';q\rangle_r \langle b+b';q\rangle_{2r}} x^{2r} \frac{q^{-rb'+ra+\binom{r}{2}}}{(xq^{-b'};q)_{a+r}} = \\
&\frac{1}{(xq^{-b'};q)_a} \sum_{r=0}^{\infty} \frac{(-1)^r \langle a,b,b',b+b'-a;q\rangle_r}{\langle 1,b+b',\frac{b+b'}{2},\frac{b+b'+1}{2},\widetilde{\frac{b+b'}{2}},\widetilde{\frac{b+b'+1}{2}};q\rangle_r} \frac{x^{2r} q^{-rb'+ra+\binom{r}{2}}}{(xq^{a-b'};q)_r} = \\
&\frac{1}{(xq^{-b'};q)_a}\, {}_6\phi_6 \left[\begin{matrix} a,b,b',b+b'-a,\infty,\infty \\ b+b',\frac{b+b'}{2},\frac{b+b'+1}{2},\widetilde{\frac{b+b'}{2}},\widetilde{\frac{b+b'+1}{2}} \end{matrix} |q, x^2 q^{a-b'} || \begin{matrix} - \\ xq^{a-b'} \end{matrix} \right].
\end{aligned}
$$

(64)
$$
\begin{aligned}
&{}_2\phi_1(a,\tfrac{c+1}{2};c|q,x)\,{}_2\phi_1(a,\tfrac{c-1}{2};c|q,xq^{\frac{1-c}{2}}) = \\
&\sum_{r=0}^{\infty} \frac{(-1)^r \langle a,c-a,\frac{c+1}{2},\frac{c-1}{2};q\rangle_r\ x^{2r} q^{r\frac{1-c}{2}+ra+\binom{r}{2}}}{\langle 1,c;q\rangle_r \langle c;q\rangle_{2r} (xq^{\frac{1-c}{2}};q)_{a+r}} = \\
&\frac{1}{(xq^{\frac{1-c}{2}};q)_a}\, {}_5\phi_5 \left[\begin{matrix} a,c-a,\underset{\sim}{\frac{c-1}{2}},\infty,\infty \\ c,\frac{c}{2},\frac{c+1}{2},\widetilde{\frac{c}{2}} \end{matrix} |q, x^2 q^{a+\frac{1-c}{2}} || \begin{matrix} - \\ xq^{a+\frac{1-c}{2}} \end{matrix} \right].
\end{aligned}
$$

(65)
$$
\begin{aligned}
&{}_2\phi_1(a,\tfrac{c}{2};c|q,x){}_2\phi_1(a,\tfrac{c}{2};c|q,xq^{-\frac{c}{2}})\\
&=\sum_{r=0}^{\infty}\frac{(-1)^r\langle a,c-a,\frac{c}{2},\frac{c}{2};q\rangle_r x^{2r}q^{-r\frac{c}{2}+ra+\binom{r}{2}}}{\langle 1,c;q\rangle_r\langle c;q\rangle_{2r}(xq^{-\frac{c}{2}};q)_{a+r}}=\\
&\frac{1}{(xq^{\frac{-c}{2}};q)_a}{}_5\phi_5\left[\begin{matrix}a,c-a,\frac{c}{2},\infty,\infty\\ c,\frac{c+1}{2},\widetilde{\frac{c}{2}},\widetilde{\frac{c+1}{2}}\end{matrix}|q,x^2q^{a-\frac{c}{2}}||\begin{matrix}-\\ xq^{a-\frac{c}{2}}\end{matrix}\right].
\end{aligned}
$$

Let $b=b'$, put $y=xq^{-b'}$ in (56), and use (60) to obtain

(66)
$$
\begin{aligned}
&{}_2\phi_1(a,b;c|q,x(1\oplus_q q^{-b}))=\sum_{r=0}^{\infty}\frac{\langle b;q\rangle_r\langle a;q\rangle_{2r}}{\langle 1;q\rangle_r\langle c;q\rangle_{2r}}x^{2r}q^{r(r-1)}\\
&\times{}_2\phi_1(a+2r,2b+2r;c+2r|q,xq^{-b}).
\end{aligned}
$$

We can easily find q-analogues of some further expansions in Burchnall-Chaundy I.

Theorem 3.9. *A q-analogue of* [9, p.256 (44)].

(67)
$$
\begin{aligned}
&{}_2\phi_1(a,b;c|q,(x\oplus_q y\boxminus_q xy)=\\
&\sum_{r=0}^{\infty}\frac{(-xy)^r\langle a,b;q\rangle_r}{\langle 1,c;q\rangle_r}q^{\binom{r}{2}}\;{}_2\phi_1(a+r,b+r;c+r|q,q^{-r}(x\oplus_q y)).
\end{aligned}
$$

Proof. This can also be written in the form

(68)
$$
\begin{aligned}
&\sum_{r=0}^{\infty}(-1)^r\frac{\langle a,b,;q\rangle_r x^r y^r q^{\binom{r}{2}}}{\langle 1,c;q\rangle_r}\sum_{k=0}^{\infty}\frac{\langle a+r,b+r;q\rangle_k}{\langle 1,c+r;q\rangle_k}\sum_{s=0}^{k}\binom{k}{s}_q x^s y^{k-s}=\\
&\sum_{l=0}^{\infty}\frac{\langle a,b;q\rangle_l}{\langle 1,c;q\rangle_l}\sum_{t=0}^{l}x^t y^{l-t}\binom{l}{t}_q\sum_{m=0}^{\infty}\frac{\langle 1-c-l,-t,t-l;q\rangle_m}{\langle 1,1-a-l,1-b-l;q\rangle_m}q^{m(-a+1-b+c)}.
\end{aligned}
$$

(69)
$$
\begin{aligned}
&LHS=\sum_{r,k=0}^{\infty}\sum_{s=0}^{k}(-1)^r\frac{\langle a,b;q\rangle_{r+k}}{\langle 1;q\rangle_r\langle c;q\rangle_{r+k}}\frac{x^{r+s}y^{r+k-s}q^{\binom{r}{2}}}{\langle 1;q\rangle_{k-s}\langle 1;q\rangle_s}.\\
&RHS=\sum_{m,l=0}^{\infty}\sum_{t=0}^{l}\frac{\langle a,b;q\rangle_{l-m}}{\langle c;q\rangle_{l-m}}\frac{\langle -t,t-l;q\rangle_m x^t y^{l-t}(-1)^m q^{-\binom{m}{2}}}{\langle 1;q\rangle_t\langle 1;q\rangle_m\langle 1;q\rangle_{l-t}}q^{ml}=\\
&\sum_{m,l=0}^{\infty}\sum_{t=0}^{l}\frac{\langle a,b;q\rangle_{l-m}}{\langle c;q\rangle_{l-m}}\frac{x^t y^{l-t}(-1)^m q^{\binom{m}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_{t-m}\langle 1;q\rangle_{l-t-m}}.
\end{aligned}
$$

□

We now show that the previous proof can be made in a more systematic way. We can write the two first q-Taylor's series in the forms

$$
{}_2\phi_1(a,b;c|q,(x\ominus_q h)=\sum_{r=0}^{\infty}\frac{(-h)^r\langle a,b;q\rangle_r}{\langle 1,c;q\rangle_r}\;{}_2\phi_1(a+r,b+r;c+r|q,x)=
\sum_{r=0}^{\infty}\frac{h^r x^{-r}\langle -\theta_1;q\rangle_r q^{-\binom{r}{2}}}{\langle 1;q\rangle_r}\;{}_2\phi_1(a,b;c|q,xq^r). \tag{70}
$$

$$
{}_2\phi_1(a,b;c|q,(x\boxminus_q h)=\sum_{r=0}^{\infty}\frac{(-h)^r\langle a,b;q\rangle_r q^{\binom{r}{2}}}{\langle 1,c;q\rangle_r}\times
{}_2\phi_1(a+r,b+r;c+r|q,x)=\sum_{r=0}^{\infty}\frac{h^r x^{-r}\langle -\theta_1;q\rangle_r}{\langle 1;q\rangle_r}\;{}_2\phi_1(a,b;c|q,xq^r). \tag{71}
$$

The following two formulas occurred in slightly different form in [26, p. 78].

By replacing the function arguments x and h by $x\oplus_q y,\ xy$ we get the following second q-analogue of [9, (44) p. 256].

$$
{}_2\phi_1(a,b;c|q,(x\oplus_q y\ominus_q xy)=
\sum_{r=0}^{\infty}\frac{(-xy)^r\langle a,b;q\rangle_r}{\langle 1,c;q\rangle_r}\;{}_2\phi_1(a+r,b+r;c+r|q,x\oplus_q y). \tag{72}
$$

We can rewrite (72) symbolically as

$$
{}_2\phi_1(a,b;c|q,(x\oplus_q y\ominus_q xy)=
{}_4\phi_2\left[\begin{matrix}1-c-\theta_1-\theta_2,-\theta_1,-\theta_2,\infty\\ 1-a-\theta_1-\theta_2,1-b-\theta_1-\theta_2\end{matrix}\;|q,-q^{-a-b+c+1}\right]
{}_2\phi_1(a,b;c|q,(x\oplus_q y). \tag{73}
$$

By replacing the function arguments x and h by $x\oplus_q y\ominus_q xy,\ -xy$, we get the following q-analogue of [9, (45) p. 256].

$$
{}_2\phi_1(a,b;c|q,(x\oplus_q y)=
\sum_{r=0}^{\infty}\frac{(xy)^r\langle a,b;q\rangle_r q^{\binom{r}{2}}}{\langle 1,c;q\rangle_r}\;{}_2\phi_1(a+r,b+r;c+r|q,x\oplus_q y\ominus_q xy). \tag{74}
$$

According to Jackson [26, p. 78], it is not possible to find q-analogues of the formulas [9, (46)-(51) p. 256].

4. An expansion formula for a $\Phi^{2:1}_{2:0}$ function

We now turn to a different problem, which can be solved by a similar technique. By Jackson's theorem we obtain the following generalization of certain expansions in [19].

$$\begin{aligned}&\bigtriangledown_q(h_1)\bigtriangledown_q(h_2)\bigtriangleup_q(h_3)\bigtriangleup_q(h_4)=\\&\sum_{k=0}^{\infty}\frac{\langle h_4-1,h_4-h_1,h_4-h_2,-\theta_1,-\theta_2,\frac{h_4+1}{2},\widetilde{\frac{h_4+1}{2}},e;q\rangle_k}{\langle 1,h_1,h_2,h_4+\theta_1,h_4+\theta_2,h_4-e,\frac{h_4-1}{2},\widetilde{\frac{h_4-1}{2}};q\rangle_k}q^k,\end{aligned}\tag{75}$$

$$e=-1+\theta_1+\theta_2+h_1+h_2,\quad h_1+h_2=h_3+h_4.$$

This implies the following theorem:

Theorem 4.1.

(76)
$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{\langle a,b;q\rangle_{m+n}\langle a+b-c;q\rangle_m\langle a+b-c;q\rangle_n}{\langle c,a+b-c;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1;q\rangle_n}x^my^n=\\&\sum_{r,m,n=0}^{\infty}\frac{\langle c-1,c-a,c-b,a,b,\frac{c+1}{2},\widetilde{\frac{c+1}{2}},a+b+2r+m+n-1;q\rangle_r}{\langle 1,c,c,c+r,c+r,a+b+r+m+n-c,\frac{c-1}{2},\widetilde{\frac{c-1}{2}};q\rangle_r}\times\\&\frac{\langle a+r,b+r;q\rangle_m\langle a+r,b+r;q\rangle_n}{\langle 1,c+2r;q\rangle_m\langle 1,c+2r;q\rangle_n}(-1)^rx^{r+m}y^{r+n}q^{\binom{r}{2}+r(a+b-c)}.\end{aligned}$$

Proof.

(77)

$$\bigtriangledown_q(a)\bigtriangledown_q(b)\bigtriangleup_q(c)\bigtriangleup_q(a+b-c)\ {}_2\phi_1(a,b;c|q,x)\ {}_2\phi_1(a,b;c|q,y)=$$

$$\sum_{r=0}^{\infty}\frac{\langle c-1,c-a,c-b,-\theta_1,-\theta_2,\frac{c+1}{2},\widetilde{\frac{c+1}{2}},a+b+\theta_1+\theta_2-1;q\rangle_r}{\langle 1,a,b,c+\theta_1,c+\theta_2,c+1-a-b-\theta_1-\theta_2,\frac{c-1}{2},\widetilde{\frac{c-1}{2}};q\rangle_r}q^r$$

$$\sum_{m,n=0}^{\infty}\frac{\langle a,b;q\rangle_m\langle a,b;q\rangle_n}{\langle 1,c;q\rangle_m\langle 1,c;q\rangle_n}x^m y^n=$$

$$\sum_{r=0}^{\infty}\frac{\langle c-1,c-a,c-b,-\theta_1,-\theta_2,\frac{c+1}{2},\widetilde{\frac{c+1}{2}},a+b+\theta_1+\theta_2-1;q\rangle_r}{\langle 1,a,b,c,c,c+1-a-b-\theta_1-\theta_2,\frac{c-1}{2},\widetilde{\frac{c-1}{2}};q\rangle_r}x^r y^r q^r$$

$$\sum_{m,n=0}^{\infty}\frac{\langle a,b;q\rangle_m\langle a,b;q\rangle_n}{\langle 1,c+r;q\rangle_m\langle 1,c+r;q\rangle_n}x^m y^n=$$

$$\sum_{r=0}^{\infty}\frac{\langle c-1,c-a,c-b,a,a,b,b,\frac{c+1}{2},\widetilde{\frac{c+1}{2}},a+b+\theta_1+\theta_2-1;q\rangle_r}{\langle 1,a,b,c+1-a-b-\theta_1-\theta_2,\frac{c-1}{2},\widetilde{\frac{c-1}{2}};q\rangle_r\langle c,c;q\rangle_{2r}}x^r y^r$$

$$\sum_{m,n=0}^{\infty}\frac{\langle a+r,b+r;q\rangle_m\langle a+r,b+r;q\rangle_n}{\langle 1,c+2r;q\rangle_m\langle 1,c+2r;q\rangle_n}x^m y^n q^{-r(r+m+n)}=$$

$$\sum_{r=0}^{\infty}\frac{\langle c-1,c-a,c-b,a,b,\frac{c+1}{2},\widetilde{\frac{c+1}{2}},a+b+2r+m+n-1;q\rangle_r}{\langle 1,a+b+r+m+n-c,\frac{c-1}{2},\widetilde{\frac{c-1}{2}};q\rangle_r\langle c,c;q\rangle_{2r}}x^r y^r$$

$$\sum_{m,n=0}^{\infty}\frac{\langle a+r,b+r;q\rangle_m\langle a+r,b+r;q\rangle_n}{\langle 1,c+2r;q\rangle_m\langle 1,c+2r;q\rangle_n}x^m y^n q^{\binom{r}{2}+r(a+b-c)}(-1)^r.$$

□

The left hand side can also be written in the form

$$\Phi^{2:1}_{2:0}\left[\begin{array}{c} a,b:a+b-c;a+b-c \\ c,a+b-c:- \end{array}|q;x,y\right] \tag{78}$$

This implies the following identity:

(79)

$$\frac{\langle a,b;q\rangle_{m+n}\langle a+b-c;q\rangle_m\langle a+b-c;q\rangle_n}{\langle c,a+b-c;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1;q\rangle_n}=$$
$$\sum_{r=0}^{min(m,n)}\frac{\langle c-1,c-a,c-b,a,b,\frac{c+1}{2},\widetilde{\frac{c+1}{2}},a+b+m+n-1;q\rangle_r}{\langle 1,a+b-r+m+n-c,\frac{c-1}{2},\widetilde{\frac{c-1}{2}};q\rangle_r\langle c,c;q\rangle_{2r}}\times$$
$$\frac{\langle a+r,b+r;q\rangle_{m-r}\langle a+r,b+r;q\rangle_{n-r}}{\langle 1,c+2r;q\rangle_{m-r}\langle 1,c+2r;q\rangle_{n-r}}q^{\binom{r}{2}+r(a+b-c)}(-1)^r.$$

In [19] we gave some q–analogues of the Appell and Kampé de Fériet [7, p. 24-25], where we assumed that the absolute value of the function arguments are small. Here is another one.

Theorem 4.2. *A q–analogue of* [7, (28), p. 24]

(80)
$$\Phi_1(\alpha;\beta,\beta';\beta+\beta'|q;x_1,x_2)\cong\frac{1}{(x_2;q)_\alpha}$$
$${}_2\phi_2(\alpha,\beta;\beta+\beta'|q,-x_1\oplus_{q,s}x_2q^{\beta'}||-;x_2q^\alpha),$$

where this q-subtraction is defined by

(81)
$$(-x_1\oplus_{q,s}x_2)^n\equiv\sum_{k=0}^{n}\binom{n}{k}_q(-x_1)^kx_2^{n-k}q^{\frac{k^2+k-2nk}{2}},\ n=0,1,2,\ldots.$$

The symbol $\cong$ denotes that the equality is purely formal.

5. Conclusion

The method to use the tilde operator to obtain q–analogues of hypergeometric function arguments expressed as $x\times 2^{\pm n}$ was first presented in [19]. In the present paper, this approach was used in formulas (4),(8)-(9), (10), (63)- (65), (75). In [19] we also presented the generalized definition of q–hypergeometric series (28), which elucidates the integration property of q–calculus. This definition is used in the formulas (4) and (10) of the present paper. The Heine notation ∞ has been used several times, like in the definition of two of the q-trigonometric functions.

6. Appendix

There are two improvements of [19]. First [19, Corollary 3.8 p, 218] a q–analogue of a reduction formula for the Humbert function from

Srivastava [35, (13), p.97], [19, Corollary 3.8 p, 218] should be changed to

$$\sum_{m,n=0}^{\infty} \frac{(-1)^n \langle \mu; q\rangle_{m+n} x^{m+n} q^{-mn}}{\langle 1,\nu; q\rangle_m \langle 1,\nu; q\rangle_n} =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \langle \widetilde{-\nu+1}-2n; q\rangle_n x^{2n} q^{\binom{n}{2}+n\nu}}{\langle 1,\widetilde{1},\nu; q\rangle_n} \frac{\langle \frac{\mu}{2}, \frac{\mu+1}{2}, \widetilde{\frac{\mu}{2}}, \widetilde{\frac{\mu+1}{2}}; q\rangle_n}{\langle \frac{\nu}{2}, \frac{\nu+1}{2}, \widetilde{\frac{\nu}{2}}, \widetilde{\frac{\nu+1}{2}}; q\rangle_n} \equiv \tag{82}$$

$${}_6\phi_{6,\ k}\left[\begin{matrix} \frac{\mu}{2}, \frac{\mu+1}{2}, \widetilde{\frac{\mu}{2}}, \widetilde{\frac{\mu+1}{2}}, \infty \\ \frac{\nu}{2}, \frac{\nu+1}{2}, \widetilde{\frac{\nu}{2}}, \widetilde{\frac{\nu+1}{2}}, \nu, \widetilde{1} \end{matrix} |q, -x^2 q^{\nu}|| - ||| \begin{matrix} \langle -\nu+\widetilde{1}-2k; q\rangle_k \\ - \end{matrix}\right].$$

The following q–analogue of Karlsson [30, 2.5, p. 201], [19, 90, p. 224] can be written in a much more concise form as

$$\begin{gathered} \Phi_D^{(3)}(a+1, 1+a-c, b, b; c|q; -q^{c-a}, x, -x) = \Gamma_q \left[\begin{matrix} c, \frac{a}{2} \\ a, c-\frac{a}{2} \end{matrix}\right] \times \\ \frac{\langle \widetilde{1}, \widetilde{c-\frac{a}{2}}; q\rangle_\infty}{\langle \widetilde{c-a}, \widetilde{1+\frac{a}{2}}; q\rangle_\infty (1+q^{\frac{a}{2}})}\ {}_2\phi_1\left[\begin{matrix} b, \frac{a}{2} \\ c-\frac{a}{2} \end{matrix} |q^2, x^2\right]. \end{gathered} \tag{83}$$

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