ON THE THEORY OF THE Γ_q FUNCTION

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Abstract. We consider the Γ_q function for $0 < |q| < 1$ and complex function values. q-Analogues of Euler's constant, the Gaussian Ψ function, the Euler and Weierstrass formulas for $\Gamma(z)$ are introduced. The meromorphic continuation of the $\Gamma_q$function is found. For the q-Riemann zeta function [26], we show a multiplication formula with the Γ_q function. The Jacobi elliptic functions sn u, cn u and dn u may be expressed in the form $\sin x$, $\cos x$ and 1 times a balanced Γ_q function. We give a solution of the Truesdell [40] F_q equation.

1. Introduction

In this article we will consider the Γ_q function for $0 < |q| < 1$ and complex z. This treatment of Γ_q is new and the purpose is to use the full strength of complex analysis to find q-analogues of fundamental formulas for the Γ function. The introduction of the q-Euler constant leads to a q-analogue of the Weierstrass product formula for Γ. We will show that the Jacobi elliptic functions can be expressed as balanced Γ_q-functions.

We now make a comparison between the historical developments of the Fakultäten and the Γ_q-function, which have developed parallel with each other. Euler studied the Γ-function for integer values and found many formulas that today bear his name. Eulers contemporary Stirling studied series of the form

$$f(x) = \frac{a}{x(x+1)} + \frac{b}{x(x+1)(x+2)} + \frac{c}{x(x+1)(x+2)(x+3)} + \dots . \quad (1)$$

Stirling also found the *Euler-Pfaff-Kummer transformation formula* for the special case $a = 1$, when solving a difference equation. [41, S. 35]. We see that the Stirling expressions have the Pochhammer symbol in the denominator. He was, however, not the first, already Wallis and Newton had made such calculations. In the period to 1820 the

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Fakultäten would develop in different directions in England and Germany. The former would strictly follow the Fluxionsnotation, like in a report by Hamilton from 1837. Therefore, we focus on the development on the continent.

Vandermonde 1772 [42] and Kramp 1798 [28] had introduced so called factorials. These objects were divided into four categories: positive, negative, whole and broken exponent. Each class had its own laws, like for the q-factorial. The *Fakultäten* were also treated by Ettingshausen and Arbogast. Vandermonde and Kramp tried to extend this function to all $x \in \mathbb{R}+$. As Bessel showed, this was not so good. Bessel tried to remedy this by defining the function in another way. In this context, Bessel [2, S. 348] proved a formula that many similarities with the Euler reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{2}$$

In the year 1824 M. Ohm and Öttinger tried to repair the theory of the *Fakultäten* without introducing complex variables and the principal branch of the logarithm. In the years 1843 and 1856 [44] Weierstrass put up serious criticism against the *Fakultäten.* The Kramp treatment does not contain universal definitions and no consideration is taken as to the convergence and divergence of the applied infinite series and products. The definition of Bessel und Ohm contains a limit, which has different values for $x > 0$ and $x < 0$. Weierstrass then put the legitimate question: *how do we extend the definition of Fakultät to complex values of x as well, without beeing advised to arbitrariness?* Today we have a similar situation: Almost all authors define the Γ_q function only for real q.

Exactly as Weierstrass predicted 1856, we will show that the complex numbers are necessary for a full understanding of the Γ_q function. We shall briefly discuss the differences in the definition of balanced Γ_q functions in comparison to previous work. In the previous work [12], a balanced Γ_q function was defined as follows:

$$\sum_{k=1}^{p} a_k = \sum_{k=1}^{r} b_k, \quad p = r. \tag{3}$$

The need for balanced Γ functions was already highlighted by Mellin [32]. In [12] we have extended this rule to the q-case. In this article we go a step further.

Theorem 1.1. *An inequality for* $\mathrm{E}_q(-x)$.

$$\mathrm{E}_q(-x) > e^{-x}, 0 < q < 1, \quad x > 0. \tag{4}$$

Theorem 1.2. *An inequality for* $\mathrm{E}_{\frac{1}{q}}(-x)$.

$$\mathrm{E}_{\frac{1}{q}}(-x) < e^{-x}, x \neq 0,\ 0 < q < 1. \tag{5}$$

Theorem 1.3. [31, p. 62] *The chain rule for derivatives works also for complex functions.*

Theorem 1.4. [34, p. 222] *The logarithm is an analytic function at every point of its Riemann surface and satisfies*

$$\mathrm{D}\log(z) = \frac{1}{z} \tag{6}$$

throughout the surface.

2. The q-Euler constant

We have

$$\log(\Gamma_q(z)) = (1-z)\log(1-q) + \sum_{k=0}^{\infty} \log(1-q^{1+k}) - \log(1-q^{z+k}). \tag{7}$$

A logarithmic differentiation gives, compare [27, 2.3 p.3]

$$\mathrm{D}\log(\Gamma_q(z)) = -\log(1-q) + \sum_{k=0}^{\infty} \frac{q^{z+k}\log q}{(1-q)\{z+k\}_q}. \tag{8}$$

Remember the definition of Euler's constant $\gamma\ \approx 0.5772156$:

$$\gamma \equiv \lim_{n\to\infty} \left(\sum_{k=1}^{n} \left(\frac{1}{k}\right) - \log(n) \right). \tag{9}$$

Definition 1. In order to find a q-analogue γ_q of Euler's constant, we put

$$\gamma_q \equiv \log(1-q) - \frac{\log q}{(1-q)} \sum_{k=1}^{\infty} \frac{q^k}{\{k\}_q}. \tag{10}$$

The slow convergence of this series is partially explained by the divergence of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Theorem 2.1.

$$\lim_{q\to 1} \gamma_q = \gamma. \tag{11}$$

Proof. This follows from $\gamma_q = -\Gamma_q'(1)$. □

Remark 1. This q-Euler constant has been implicitly defined in [29, p. 75, (2.5)], but not mentioned by name there. Krattenthaler and Srivastava only considered the Ψ_q function for real q, and made a distinction between $q < 1$ and $q > 1$. On the other hand Kurokawa and Wakayama [30, p. 937] defined a completely different q-Euler constant, which is adapted to integer values of q.

We find by (7)

$$\log(\Gamma_q(z)) = \gamma_q(1-z) + \sum_{k=0}^{\infty}\big(\log(1-q^{1+k}) - \log(1-q^{z+k})\big) + \frac{(1-z)\log q}{(1-q)}\sum_{k=1}^{\infty}\frac{q^k}{\{k\}_q}. \tag{12}$$

The following table lists some approximative values of γ_q with six decimal places. The series converges very slowly,

q	γ_q
.95	0.564484
.96	0.567068
.965	0.568353
.97	0.569633
.975	0.570908
.98	0.572179
.985	0.573445
.99	0.574707
.991	0.574958
.992	0.57521
.993	0.575461
.994	0.575712
.995	0.575963
.996	0.576214
.997	0.576465
.998	0.576715
.9981	0.57674
.9982	0.576765
.9983	0.57679
.9984	0.576815
.9985	0.57684
.9986	0.576865
.9987	0.57689
.9988	0.576915
.9989	0.576939
.999	0.576965
.9995	0.577091

We remark that the differences $\delta(\gamma_q)$ have approximately the same value 2 $\times 10^{-5}$ in the interval $[.998, .999]$, the definition of γ_q makes sence.

The derivative $\frac{\mathrm{D}\gamma_q}{dq}$ (for real q) is

$$\frac{\mathrm{D}\gamma_q}{dq} = -\frac{1}{|1-q|} - \sum_{k=1}^{\infty} \frac{1}{\{k\}_q} \left[\frac{(1-q)\left(kq^{k-1}\log q + \frac{q^k}{|q|}\right) + q^k \log q}{(1-q)^2} \right]. \tag{13}$$

The function $F(q) \equiv \frac{\mathrm{D}\gamma_q}{dq}$ converges very slowly.

We have the following values for the partial sums $F(.9989, m)$:

m	F(.9989,m)
100	-814.297
1000	-153.88
1500	79.6181
1700	152.798
8000	581.591
10000	582.747
12000	582.905
14000	582.925
16000	582.928

Definition 2. The Gauss Ψ function has the following q-analogue:

$$\Psi_q(z) \equiv \mathrm{D}\log(\Gamma_q(z+1)). \tag{14}$$

This implies by (8)

Theorem 2.2.

$$\begin{aligned}\Psi_q(z) &= -\gamma_q + \sum_{k=1}^{\infty} \frac{q^k \log q}{1-q}\left(\frac{q^z}{\{z+k\}_q} - \frac{1}{\{k\}_q}\right)\\ &= -\gamma_q - \sum_{k=1}^{\infty} \frac{q^k \log q \{z\}_q}{(1-q)\{k\}_q\{z+k\}_q}.\end{aligned} \tag{15}$$

Die Gaußsche Ψ-Funktion kann auch in der Form von Lambertscher Reihe geschrieben werden:

$$\Psi_q(z) = -\gamma_q - \log q \frac{1-q^z}{(1-q)(1-q^{z+1})}\ {}_3\phi_2(1, z+1, \infty; 2, z+2|q;q). \tag{16}$$

The derivative of this is [38, S. 351], [1, S. 128, 3.4]*:*

$$\mathrm{D}\Psi_q(z) = \sum_{k=1}^{\infty} \frac{q^{-k-z}(\log q)^2}{(q^{-k-z}-1)^2} \equiv \frac{(\log q)^2}{(1-q)^2} \sum_{k=1}^{\infty} \frac{q^{k+z}}{(\{z+k\}_q)^2}. \tag{17}$$

According to the q-Bohr-Mollerup theorem [1, S. 128] *we have*

$$\mathrm{D}\Psi_q(x) > 0, x > -1. \tag{18}$$

The function $\mathrm{D}\Psi_q(z)$ has simple poles at $x = -n$.

The special case

$$\mathrm{D}\Psi_q(0) = \sum_{k=1}^{\infty} \frac{q^k(\log q)^2}{(1-q)^2} \frac{1}{(\{k\}_q)^2} \tag{19}$$

is a q-analogue of $\frac{\pi^2}{6}$.

Higher derivatives of Ψ are:

$$\mathrm{D}^2\Psi_q(z) = (\log q)^3 \sum_{k=1}^{\infty} \frac{q^{k+z}(1+q^{z+k})}{(1-q^{z+k})^3}. \tag{20}$$

The special case

$$\mathrm{D}^2\Psi_q(0) = (\log q)^3 \sum_{k=1}^{\infty} \frac{q^{k}(1+q^{k})}{(1-q^{k})^3} \tag{21}$$

is a q-analogue of $-2\zeta(3)$, where ζ denotes the Riemann Zeta function.

$$\mathrm{D}^3\Psi_q(z) = (\log q)^4 \sum_{k=1}^{\infty} \frac{q^{z+k}+4q^{2z+2k}+q^{3z+3k}}{(1-q^{z+k})^4}. \tag{22}$$

The special case

$$\mathrm{D}^3\Psi_q(0) = (\log q)^4 \sum_{k=1}^{\infty} \frac{q^{k}+4q^{2k}+q^{3k}}{(1-q^{k})^4} \tag{23}$$

is a q-analogue of $\frac{\pi^4}{15}$. All special values of the Ψ derivatives can be found in [37, S. 627].

Proof. We start with (20).

$$\mathrm{D}^2\Psi_q(z) = (\log q)^2 D \sum_{k=1}^{\infty} \frac{e^{(k+z)\log q}}{(1-e^{(k+z)\log q})^2}$$
$$= (\log q)^2 \sum_{k=1}^{\infty} \frac{(\log q)q^{k+z}(1-q^{z+k})^2 + 2q^{k+z}(1-q^{z+k})(\log q)q^{k+z}}{(1-q^{z+k})^4} = RHS. \tag{24}$$

Formula (22) is proved in a similar way.

$$\mathrm{D}^3\Psi_q(z) = (\log q)^3 D \sum_{k=1}^{\infty} \frac{e^{(k+z)\log q}+e^{2(k+z)\log q}}{(1-e^{(k+z)\log q})^3} = (\log q)^3 \sum_{k=1}^{\infty}$$
$$\frac{(\log q)(q^{k+z}+2q^{2k+2z})(1-q^{z+k})^3 + 3(q^{k+z}+q^{2k+2z})q^{k+z}(1-q^{z+k})^2(\log q)q^{k+z}}{(1-q^{z+k})^4}$$
$$= RHS. \tag{25}$$

□

3. q-ANALOGUES OF WELL-KNOWN FORMULAS FOR THE Γ_q FUNCTION

Our next task is to try to find a q-analogue of Euler's formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n+1)^z n!}{(z)_{n+1}}. \tag{26}$$

The following integral representation is known from [6].

$$\Gamma_q(z) = \int_0^{\frac{1}{1-q}} t^{z-1} \mathrm{E}_{\frac{1}{q}}(-qt)\, d_q(t),\ \Re(z) > 0. \tag{27}$$

We have the following limit form at our disposal:

$$\Gamma_q(z) = \lim_{n\to\infty} \int_0^{\frac{1}{1-q}} t^{z-1} (qt(1-q);q)_n\, d_q(t),\ \Re(z) > 0. \tag{28}$$

Furthermore,

$$\mathrm{D}_q \frac{1}{(z;q)_\alpha} = \frac{\{\alpha\}_q}{(z;q)_{\alpha+1}}. \tag{29}$$

q-integration by parts gives

$$\Gamma_{n,q}(z) = \int_0^{\frac{1}{1-q}} t^{z-1}(qt(1-q);q)_n\, d_q(t) = \left[\frac{t^z(t(1-q);q)_n}{\{z\}_q}\right]_0^{\frac{1}{1-q}} - \frac{\{-n\}_q(1-q)q^n}{\{z\}_q} \int_0^{\frac{1}{1-q}} t^z (qt(1-q);q)_{n-1} = \frac{\{n\}_q(1-q)}{\{z\}_q} \Gamma_{n-1,q}(z+1). \tag{30}$$

The initial value is

$$\Gamma_{1,q}(z) = \int_0^{\frac{1}{1-q}} t^{z-1}(qt(1-q);q)_n\, d_q(t) = \frac{(1-q)^{-z}}{\{z\}_q\{z+1\}_q}. \tag{31}$$

We find that the q-analogue of Euler's limit formula (26) is

$$\Gamma_q(z) = \lim_{n\to\infty} \frac{\{n\}_q!(1-q)^{-z}}{\{z\}_{n+1,q}}. \tag{32}$$

Formula (32) is equivalent to the defining formula for $\Gamma_q(z)$.

Exponentiation of (12) gives a q-analogue of the Weierstrass formula for $\Gamma(z)$.

$$\frac{1}{\Gamma_q(z)} = \frac{e^{\gamma_q z}\langle z;q\rangle_\infty}{e^{\gamma_q}\langle 1;q\rangle_\infty} \prod_{k=1}^{\infty} \exp\left(\frac{q^k(z-1)\log q}{(1-q)\{k\}_q}\right),\ z\in\mathbb{C}. \tag{33}$$

The poles of $\Gamma_q(z)$, $z = -n \pm \frac{2k\pi i}{\log q},\ n,k\in\mathbb{N}$ are contained in the factor $\langle z;q\rangle_\infty$. The factor $\prod_{k=1}^{\infty} e^{-\frac{z}{k}}$ is contained in the last factor.

Formula (33) implies

$$\frac{e^{\gamma_q z}}{e^{\gamma_q}} \prod_{k=1}^{\infty} \exp\left(\frac{q^k (z-1)\log q}{(1-q)\{k\}_q}\right) = (1-q)^{z-1}. \tag{34}$$

Jackson [21, p. 65] has given another q-analogue of the Weierstrass formula for $\Gamma(z)$. Formula (33) together with its proof is however superior.

In the classical case we have the following formula

$$\mathrm{D}\log(\Gamma(z)\Gamma(1-z)) = -\frac{\pi}{\tan \pi z}. \tag{35}$$

Theorem 3.1. *A q-Analogue of (35).*

$$\mathrm{D}\log(\Gamma_q(z)\Gamma_q(1-z)) = \frac{\log q}{2}\left(-\coth(\frac{z}{2}\log q) + \sum_{m=0}^{\infty}\frac{q^{1+\frac{z}{2}+m}}{1-q^{1+\frac{z}{2}+m}} - \sum_{m=0}^{\infty}\frac{q^{1-\frac{z}{2}+m}}{1-q^{1-\frac{z}{2}+m}}\right). \tag{36}$$

Proof.

$$\log(\Gamma_q(z)\Gamma_q(1-z)) = \log\left(\frac{i\mathrm{QE}(\frac{1}{8})(1-q)(\langle 1;q\rangle_\infty)^3}{\mathrm{QE}(\frac{z}{2})\theta_1(\log \mathrm{QE}(\frac{-iz}{2}), \mathrm{QE}(\frac{1}{2}))}\right) = \log(f(q)) - \frac{z}{2}\log q - \log(e^{\frac{z}{2}\log q} - e^{-\frac{z}{2}\log q}) - \sum_{m=0}^{\infty}\log(1-e^{\log q(1+\frac{z}{2}+m)}) - \sum_{m=0}^{\infty}\log(1-e^{\log q(1-\frac{z}{2}+m)}), \tag{37}$$

where $f(q)$ is a function independent of z.

The derivative of this expression gives the result. □

We can find a similar q-integral formula for $\Gamma_q(z)$ (compare [24, p. 199 (16)]):

Theorem 3.2.

$$\Gamma_q(z) = \int_0^{\infty} t^{z-1}\mathrm{E}_{\frac{1}{q}}(-qt)\, d_q(t) \quad \Re(z) > 0 \tag{38}$$

Proof. q-Integration by parts gives

$$\int_0^\infty t^{z-1}\mathrm{E}_{\frac{1}{q}}(-qt)\,d_q(t) = \Big[-t^{z-1}\mathrm{E}_{\frac{1}{q}}(-t)\Big]_0^\infty + \{z-1\}_q \int_0^\infty t^{z-2}\mathrm{E}_{\frac{1}{q}}(-qt)\,d_q(t). \tag{39}$$

The first term on the right is zero because the zeros of $\mathrm{E}_{\frac{1}{q}}(-t)$ approach ∞ quicker than any polynomial. We obtain the required recurrence: $\Gamma_q(z) = \{z-1\}_q\Gamma_q(z-1)$. □

The following formula is proved in the same way.

$$\Gamma_q(z) = q^{-\binom{z}{2}} \int_0^\infty t^{z-1}\mathrm{E}_q(-t)\,d_q(t). \tag{40}$$

Theorem 3.3. *A q-analogue of* [36, S. 689], [35, S. 143].

The Γ_q-Funktion has the following meromorphic continuation into $\mathbb{C}$:

$$\Gamma_q(z) = \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n^2+n}{2}}}{\{z+n\}_q\{n\}_q!} + \int_1^{\frac{1}{1-q}} t^{z-1}\mathrm{E}_{\frac{1}{q}}(-qt)\;d_q(t). \tag{41}$$

Proof.

$$\Gamma_q(z) = \int_0^1 t^{z-1}\sum_{n=0}^\infty \frac{(-t)^n q^{\frac{n^2+n}{2}}}{\{n\}_q!}\,d_q(t) + \int_1^{\frac{1}{1-q}} t^{z-1}\mathrm{E}_{\frac{1}{q}}(-qt)\,d_q(t) = \sum_{n=0}^\infty \frac{(-1)^n q^{\frac{n^2+n}{2}}}{\{z+n\}_q\{n\}_q!} + \int_1^{\frac{1}{1-q}} t^{z-1}\mathrm{E}_{\frac{1}{q}}(-qt)\;d_q(t). \tag{42}$$

□

Pringsheim [36, S. 689] writes this formula

$$\Gamma(z) = F(z) + G(z), \tag{43}$$

where $F(z) \equiv \sum_{n=0}^\infty \frac{(-1)^n}{n!(z+n)}$ and $G(z)$ denotes an entire function. A similar computation for $\mathrm{E}_q(-t)$ with the help of (40) is less useful because of the inequality (4).

Our next aim is to show that the Γ_q function has a completely different integral representation, which is given by the function $I(z,q)$.

Definition 3. The function $I(z,q)$ is given by

$$I(z,q) \equiv q^{2z-2}\frac{\{2\}_q^{z-1}}{\langle\widetilde{2};q\rangle_{z-2}}\int_0^\infty u^{2z-1}\mathrm{E}_{2,\frac{1}{q}}(-q^2u^2)\,d_q(u),\;\Re(z) > 0. \tag{44}$$

Theorem 3.4.

$$I(z,q)=\{z-1\}_q I(z-1,q). \tag{45}$$

Proof. q-Integration by parts gives

$$V(z,q)\equiv I(z,q)q^{-2z+2}\frac{\langle\widetilde{2};q\rangle_{z-2}}{\{2\}_q^{z-2}}\equiv\{2\}_q\int_0^\infty u^{2z-1}\mathrm{E}_{2,\frac{1}{q}}(-q^2u^2)\,d_q(u)=$$
$$-\left[q^{-2}u^{2z-2}\mathrm{E}_{2,\frac{1}{q}}(-u^2)\right]_0^\infty+\{2\}_q\{z-1\}_q q^{-2}\frac{1+q^{z-1}}{1+q}\times$$
$$\int_0^\infty u^{2z-3}\mathrm{E}_{2,\frac{1}{q}}(-q^2u^2)\,d_q(u). \tag{46}$$

□

By combining equations (44) and (38) we obtain a first example of a nonlinear substitution in a q-integral.

$$\int_0^\infty t^{z-1}\mathrm{E}_{\frac{1}{q}}(-qt)\,d_q(t)=q^{2z-2}\frac{\{2\}_q^{z-1}}{\langle\widetilde{2};q\rangle_{z-2}}\int_0^\infty u^{2z-1}\mathrm{E}_{2,\frac{1}{q}}(-q^2u^2)\,d_q(u). \tag{47}$$

We will now connect our results to the q-Riemann zeta function from [26]. We remark that other q-Riemann zeta functions can be found in the literature, but we prefer this one. Our q-Riemann Zeta function has the same poles as the Γ_q function.

Definition 4. Let $\mathrm{E}^\star$ be the punctured unit circle and

$$\mathrm{F}\equiv\{a+\frac{2\pi ib}{\log q}|a,b\in\mathbb{Z},a\leq 0\},\mathrm{D}\equiv\mathrm{E}^\star\times(\mathbb{C}\backslash(\mathrm{F})). \tag{48}$$

Denote the meromorphic functions in D by $\mathcal{M}er(\mathrm{D})$.

Theorem 3.5. $\mathcal{M}er(\mathrm{D})$ *is a field.*

Definition 5. [26] The q-Riemann zeta function is defined by

$$\zeta_q(z)\equiv\sum_{n=1}^\infty\frac{q^{n(z-1)}}{\{n\}_q^z},\quad\Re(z)>1. \tag{49}$$

Theorem 3.6. [26]

(1) *The function* $\zeta_q(z)$ *has a simple pole in* $1+\frac{2\pi i\mathbb{Z}}{\log q}$ *and in* $\{a+\frac{2\pi ib}{\log q}|a,b\in\mathbb{Z},a\leq 0\}$. *In particular,* $z=1$ *is a simple pole with residue* $\frac{q-1}{\log q}$.

(2) $\zeta_q(-m)=(1-q)^{-m}\left(\sum_{r=0}^m(-1)^r\binom{m}{r}\frac{1}{q^{m+1-r}-1}+\frac{(-1)^{m+1}}{(m+1)\log q}\right)$.

Theorem 3.7. [26] *For any* $s \in \mathbb{C},\ s \neq 1$, *we have*

$$lim_{q\to 1}\zeta_q(s) = \zeta(s). \tag{50}$$

We now make a multiplication in $\mathcal{M}er(\mathrm{D})$.

Theorem 3.8. *A q-analogue of* [20, p. 59].

$$\zeta_q(z)\Gamma_q(z) = \int_0^\infty u^{z-1}\sum_{n=1}^\infty q^{n(z-1)}\mathrm{E}_{\frac{1}{q}}(-q\{n\}_q u)\, d_q(u),\ \Re(z) > 1. \tag{51}$$

Proof. We start with the q-integral formula for $\Gamma_q(z)$.

$$\Gamma_q(z) = \int_0^\infty t^{z-1}\mathrm{E}_{\frac{1}{q}}(-qt)\ d_q(t) \tag{52}$$

Change variables to $t = \{n\}_q u$.

$$\Gamma_q(z) = \{n\}_q^z\int_0^\infty u^{z-1}\mathrm{E}_{\frac{1}{q}}(-\{n\}_q qu)\ d_q(u) \tag{53}$$

We obtain the following expression for $\zeta_q(z)$:

$$\begin{aligned}&\sum_{n=1}^\infty \frac{q^{n(z-1)}}{\{n\}_q^z} = \sum_{n=1}^\infty \frac{q^{n(z-1)}}{\Gamma_q(z)}\int_0^\infty u^{z-1}\mathrm{E}_{\frac{1}{q}}(-\{n\}_q qu)d_q(u) \overset{\text{by}(115)}{=}\\ &\frac{1}{\Gamma_q(z)}\int_0^\infty u^{z-1}\sum_{n=1}^\infty q^{n(z-1)}\mathrm{E}_{\frac{1}{q}}(-\{n\}_q qu)d_q(u).\end{aligned} \tag{54}$$

A multiplication with $\Gamma_q(z)$ completes the proof. □

The Euler mirror formula has the following q-analogue:

Theorem 3.9.

$$\Gamma_q(z)\Gamma_q(1-z) = \frac{iq^{\frac{1}{8}}(1-q)(\langle 1;q\rangle_\infty)^3}{q^{\frac{z}{2}}\theta_1(\frac{-iz}{2}\log q, \sqrt{q})}. \tag{55}$$

For real z and q in (55) is the first function value for θ_1 *in (55) imaginary. We then have*

$$\theta_1(iz,q) \equiv 2i\sum_{n=0}^\infty (-1)^n \mathrm{QE}\left((n+\frac{1}{2})^2\right)\sinh(2n+1)z. \tag{56}$$

This obviously explains the occurrence of the factor i in the numerator of (55). Denote the RHS of (55) by $F(z,q)$. *The function* $F(z,q)$ *satisfies the following equation:*

$$F(z+1,q) = -q^z F(z,q). \tag{57}$$

Proof. Use the quasiperiodicity

$$\theta_1(z+\frac{-i}{2}\log q,q)=-q^{-1}e^{-2iz}\theta_1(z,q).$$

□

$F(x,q)$ as real function has asymptotes für $x=n, n\in\mathbb{Z}$.
A numerical computation shows that

$$\lim_{n\to N}\frac{F(2n,q)}{F(n,q)}=(-1)^N\frac{1}{2}\mathrm{QE}\left(-\frac{N}{2}+\frac{3N^2}{2}\right). \tag{58}$$

The singularities occurring for finite values of the $\Gamma-$function ($z=-n$) can be removed by multiplication with $\sin(\pi z)$. We want to create a similar rule for the Γ_q-function. To this end, we conclude that a q-analogue of π is given by

$$(\Gamma_q(\frac{1}{2}))^2=\frac{iq^{\frac{1}{8}}(1-q)(\langle 1;q\rangle_\infty)^3}{q^{\frac{1}{4}}\theta_1(\frac{-i}{4}\log q,\sqrt{q})}. \tag{59}$$

Theorem 3.10. *The singularities occurring for finite values of the Γ_q-function ($z=-n\pm\frac{2m\pi i}{\log q}$) can be removed by division with*

$$\frac{\Gamma_q(z)\Gamma_q(1-z)}{(\Gamma_q(\frac{1}{2}))^2}=\frac{\{\frac{1}{2}\}_q\langle\frac{5}{4},\frac{3}{4};q\rangle_\infty}{\{z\}_q\langle 1+\frac{z}{2},1-\frac{z}{2};q\rangle_\infty}. \tag{60}$$

Proof. We need the product representation of θ_1.

$$\theta_1(z,q)=2q^{\frac{1}{4}}\sin z\langle 1,1+\frac{iz}{\log q},1-\frac{iz}{\log q};q^2\rangle_\infty. \tag{61}$$

□

4. Jacobi elliptic functions as Γ_q functions

Theorem 4.1. *The Jacobi elliptic functions can be written as generalized Γ_q functions.*

Proof. Introduce the following notation:

$$\begin{aligned}&q\equiv e^{-\pi\frac{K'}{K}};\ x\equiv\frac{u\pi}{2K};\ q'\equiv e^{-\pi\frac{K}{K'}};\ y\equiv\frac{u\pi}{2K'};\\ &q^{2a}\equiv e^{2ix};\ q'^{(2a')}\equiv e^{2iy};\ b\equiv\frac{\log(-1)}{\log q^2};\ b'\equiv\frac{\log(-1)}{\log q'^2}.\end{aligned} \tag{62}$$

We infer that

$$2b\equiv 0 \bmod \frac{2\pi i}{\log q^2};\ 2b'\equiv 0 \bmod \frac{2\pi i}{\log q'^2}. \tag{63}$$

According to Broch [3, p. 127 (3)-(5)] we have the following product expansions:

$$\text{sn } u = \frac{-Kie^{ix}}{\pi} \prod_{n=1}^{\infty} \frac{(1-q^{2n+2a})(1-q^{2n-2-2a})}{(1-q^{2n-1+2a})(1-q^{2n-1-2a})} \left(\frac{1-q^{2n-1}}{1-q^{2n}}\right)^2 \quad (64)$$

$$\text{cn } u = \frac{e^{ix}}{2} \prod_{n=1}^{\infty} \frac{(1+q^{2n-2-2a})(1+q^{2n+2a})}{(1-q^{2n-1+2a})(1-q^{2n-1-2a})} \left(\frac{1-q^{2n-1}}{1+q^{2n}}\right)^2 \quad (65)$$

$$\text{dn } u = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1-2a})(1+q^{2n-1+2a})}{(1-q^{2n-1-2a})(1-q^{2n-1+2a})} \left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^2 \quad (66)$$

Using the definition of b these equations can be rewritten in the following form:

$$\text{sn } u = \frac{-Kie^{ix}}{\pi} \frac{\langle a+1, -a, \frac{1}{2}, \frac{1}{2}; q^2\rangle_\infty}{\langle 1, 1, a+\frac{1}{2}, -a+\frac{1}{2}; q^2\rangle_\infty}. \quad (67)$$

$$\text{cn } u = \frac{e^{ix}}{2} \frac{\langle a+1+b, -a+b, \frac{1}{2}, \frac{1}{2}; q^2\rangle_\infty}{\langle a+\frac{1}{2}, -a+\frac{1}{2}, 1+b, 1+b; q^2\rangle_\infty}. \quad (68)$$

$$\text{dn } u = \frac{\langle a+\frac{1}{2}+b, -a+\frac{1}{2}+b, \frac{1}{2}, \frac{1}{2}; q^2\rangle_\infty}{\langle a+\frac{1}{2}, -a+\frac{1}{2}, \frac{1}{2}+b, \frac{1}{2}+b; q^2\rangle_\infty}. \quad (69)$$

These formulas can be rewritten as balanced Γ_q functions:

$$[25, p.145]\ \text{sn } u = \frac{-Kie^{ix}(1-e^{-2ix})}{\pi} \Gamma_{q^2}\left[\begin{array}{c} 1, 1, a+\frac{1}{2}, -a+\frac{1}{2} \\ a+1, 1-a, \frac{1}{2}, \frac{1}{2} \end{array}\right]. \quad (70)$$

$$\text{cn } u = \cos x\ \Gamma_{q^2}\left[\begin{array}{c} a+\frac{1}{2}, -a+\frac{1}{2}, 1+b, 1+b \\ \frac{1}{2}, \frac{1}{2}, a+1+b, 1-a+b \end{array}\right]. \quad (71)$$

$$\text{dn } u = \Gamma_{q^2}\left[\begin{array}{c} a+\frac{1}{2}, -a+\frac{1}{2}, \frac{1}{2}+b, \frac{1}{2}+b \\ a+\frac{1}{2}+b, -a+\frac{1}{2}+b, \frac{1}{2}, \frac{1}{2} \end{array}\right]. \quad (72)$$

The first equation can be written in the symmetric form

$$\text{sn } u = \sin x\ \Gamma_{q^2}\left[\begin{array}{c} 1+b, 1+b, a+\frac{1}{2}, -a+\frac{1}{2} \\ a+1, -a+1, \frac{1}{2}+b, \frac{1}{2}+b \end{array}\right]. \quad (73)$$

Gudermann showed that the elliptic functions can be developed in two series with basis q and q' [10], depending on the value of $\frac{K}{K'}$. The same is true for product expansions. According to Broch [3, p. 129 (3)-(5)] we have the following complementary product expansions:

$$\text{sn } u = \frac{2K'(1-q'^{-2a'})}{\pi(1+q'^{-2a'})} \prod_{n=1}^{\infty} \frac{(1-q'^{2n-2a'})(1-q'^{2n+2a'})}{(1+q'^{2n-2a'})(1+q'^{2n+2a'})} \left(\frac{1+q'^{2n}}{1-q'^{2n}}\right)^2 \quad (74)$$

$$\text{cn}\ u = \frac{2q'^{-2a'}}{1+q'^{-2a'}} \prod_{n=1}^{\infty} \frac{\left(1-q'^{2n-1-2a'}\right)\left(1-q'^{2n-1+2a'}\right)}{\left(1+q'^{2n-2a'}\right)\left(1+q'^{2n+2a'}\right)} \left(\frac{1+q'^{2n}}{1-q'^{2n-1}}\right)^2 \tag{75}$$

$$\text{dn}\ u = \frac{2q'^{-2a'}}{1+q'^{-2a'}} \prod_{n=1}^{\infty} \frac{\left(1+q'^{2n-1-2a'}\right)\left(1+q'^{2n-1+2a'}\right)}{\left(1+q'^{2n-2a'}\right)\left(1+q'^{2n+2a'}\right)} \left(\frac{1+q'^{2n}}{1+q'^{2n-1}}\right)^2 \tag{76}$$

Using the definition of b' these equations can be rewritten in the following form:

$$\text{sn}\ u = \frac{2K'(1-q'^{-2a'})}{\pi(1+q'^{-2a'})} \frac{\langle a'+1, -a'+1, 1+b', 1+b'; q'^2\rangle_\infty}{\langle 1, 1, a'+1+b', -a'+1+b'; q'^2\rangle_\infty}. \tag{77}$$

$$\text{cn}\ u = \frac{2q'^{-2a'}}{1+q'^{-2a'}} \frac{\langle a'+\frac{1}{2}, -a'+\frac{1}{2}, 1+b', 1+b'; q'^2\rangle_\infty}{\langle \frac{1}{2}, \frac{1}{2}, a'+1+b', -a'+1+b'; q'^2\rangle_\infty}. \tag{78}$$

$$\text{dn}\ u = \frac{2q'^{-2a'}}{1+q'^{-2a'}} \frac{\langle a'+\frac{1}{2}+b', -a'+\frac{1}{2}+b', 1+b', 1+b'; q'^2\rangle_\infty}{\langle a'+1+b', -a'+1+b', \frac{1}{2}+b', \frac{1}{2}+b'; q'^2\rangle_\infty}. \tag{79}$$

These balanced formulas can be rewritten with the Γ_q function as

$$\text{sn}\ u = \frac{2K'(1-q'^{-2a'})}{\pi(1+q'^{-2a'})} \Gamma_{q'^2} \left[\begin{matrix} 1, 1, a'+1+b', -a'+1+b' \\ a'+1, -a'+1, 1+b', 1+b' \end{matrix} \right]. \tag{80}$$

$$\text{cn}\ u = \frac{2q'^{-2a'}}{1+q'^{-2a'}} \Gamma_{q'^2} \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, a'+1+b', -a'+1+b' \\ a'+\frac{1}{2}, -a'+\frac{1}{2}, 1+b', 1+b' \end{matrix} \right]. \tag{81}$$

$$\text{dn}\ u = \frac{2q'^{-2a'}}{1+q'^{-2a'}} \Gamma_{q'^2} \left[\begin{matrix} a'+1+b', -a'+1+b', \frac{1}{2}+b', \frac{1}{2}+b' \\ a'+\frac{1}{2}+b', -a'+\frac{1}{2}+b', 1+b', 1+b' \end{matrix} \right]. \tag{82}$$

□

5. Equivalence between tilde operator and Γ_q function

In the article [8] we found several q-analogues of transformation- and summation formulas for multiple q-hypergeometric functions. One example was

$$\begin{aligned} &\Phi_1(\alpha; \beta, \beta'; 1+\alpha+\beta-\beta' | q; q^{1-\beta'}, -q^{1-\beta'}) = \\ &= \Gamma_q \left[\begin{matrix} 1+\alpha+\beta-\beta', 1-\beta', 1+\frac{\alpha}{2} \\ 1+\alpha, 1+\beta-\beta', 1+\frac{\alpha}{2}-\beta' \end{matrix} \right] \frac{\langle 1+\widetilde{\frac{\alpha}{2}-\beta'}, \widetilde{1}; q\rangle_\infty}{\langle \widetilde{1+\frac{\alpha}{2}}, \widetilde{1-\beta'}; q\rangle_\infty}, \end{aligned} \tag{83}$$

where $|q^{1-\beta'}| < 1$. For $q = 1$ this formula is

$$\mathrm{F}_1(\alpha;\beta,\beta';1+\alpha+\beta-\beta';1,-1) = \Gamma\left[\begin{matrix} 1+\alpha+\beta-\beta', 1-\beta', 1+\frac{\alpha}{2} \\ 1+\alpha, 1+\beta-\beta', 1+\frac{\alpha}{2}-\beta' \end{matrix}\right]. \tag{84}$$

With the notation $b \equiv \frac{\log(-1)}{\log q}$ our formula (83) can be written

$$\begin{aligned} &\Phi_1(\alpha;\beta,\beta';1+\alpha+\beta-\beta'|q;q^{1-\beta'},-q^{1-\beta'}) = \\ &= \Gamma_q\left[\begin{matrix} 1+\alpha+\beta-\beta', 1-\beta', 1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}+b, 1-\beta'+b \\ 1+\alpha, 1+\beta-\beta', 1+\frac{\alpha}{2}-\beta', 1+\frac{\alpha}{2}-\beta'+b, 1+b \end{matrix}\right] \end{aligned} \tag{85}$$

All equations with balanced, generalized Γ_q functions multiplied with balanced quotients of infinite tilde q-factorials can be written as balanced, generalized Γ_q functions.

6. The q-Truesdell functions

We now come to a slight generalization of the hypergeometric function, which has been used by Barnes, Meijer, Fox and Truesdell. Truesdell [40] has set up an equation that provides the basis for his investigations of the special functions:

$$\mathrm{D}_z F(z,\alpha) = F(z,\alpha+1). \tag{86}$$

The basis for this approach can be found in a letter from Nørlund to Mittag-Leffler 1919 [18]. Nørlund says that it is difficult to proceed with difference equations if you don't master the theory of the gamma function or the theory of Bernoulli- (and Euler) polynomials. There is a similar connection in q-analysis, but here it is about q-difference equations like (93) and Γ_q function und q-Bernoulli- (und q-Euler) polynomials [11].

Definition 6. The generalized q-hypergeometric function is defined by

$$\begin{aligned} &{}_p\mathbf{\Phi}_{p-1}\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{p-1} \end{matrix} |q,z\right] \equiv \sum_{k=0}^{\infty} \Gamma_q\left[\begin{matrix} a_1+k,\ldots,a_p+k \\ b_1+k,\ldots,b_{p-1}+k, 1+k \end{matrix}\right] z^k \\ &\equiv \Gamma_q\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{p-1},1 \end{matrix}\right] \sum_{k=0}^{\infty} \frac{\langle a_1,\ldots,a_p;q\rangle_k}{\langle 1,b_1,\ldots,b_{p-1};q\rangle_k}. \end{aligned} \tag{87}$$

In case one of the denominator parameters in a hypergeometric function is a negative integer, we can take limits of the summands with singular Γ_q values in (87) to obtain

Theorem 6.1. *A q-analogue of the corrected version of* [43, S.118]

$$
{}_2\mathbf{\Phi_1}\left[\begin{array}{c} a,b \\ -N \end{array} |q;z\right]
= z^{N+1}\frac{\Gamma_q(a+N+1)\Gamma_q(b+N+1)}{\{N+1\}_q!}\,{}_2\phi_1\left[\begin{array}{c} a+N+1,b+N+1 \\ N+2 \end{array} |q;z\right], \tag{88}
$$

$-a,-b\notin\mathbb{N}$.

Proof.

$$
LS=\sum_{k=N+1}^{\infty}\Gamma_q\left[\begin{array}{c} a+k,b+k \\ -N+k,1+k \end{array}\right]z^k= \tag{89}
$$

$$
=\sum_{m=0}^{\infty}\Gamma_q\left[\begin{array}{c} a+k+N+1,b+k+N+1 \\ 2+m+N,1+m \end{array}\right]z^{m+N+1}.
$$

$$
RS=\frac{\Gamma_q(a+N+1)\Gamma_q(b+N+1)}{\langle 1;q\rangle_{N+1}(1-q)^{-N-1}}\sum_{l=0}^{\infty}\frac{\langle a+N+1,b+N+1;q\rangle_l}{\langle 1,N+2;q\rangle_l}z^{l+N+1}
$$

$$
=\sum_{l=0}^{\infty}\frac{(1-q)^{N+1+2l}}{\langle 1;q\rangle_{N+1}}\frac{\Gamma_q(a+N+1+l)\Gamma_q(b+N+1+l)}{\langle 1,N+2;q\rangle_l}z^{l+N+1}=LS. \tag{90}
$$

□

The function (87) is called balanced if

$$
a_1+\ldots+a_p=b_1+\ldots+b_{p-1}+1. \tag{91}
$$

The q-analogue of Truesdells function [40, p. 17], is defined by

Definition 7.

$$
F(z,\alpha,q)\equiv\ {}_p\mathbf{\Phi}_{p-1}\left[\begin{array}{c} \alpha+a_1,\ldots,\alpha+a_p \\ \alpha+b_1,\ldots,\alpha+b_{p-1} \end{array} |q,z\right]. \tag{92}
$$

It satisfies a q- analogue of Truesdells equation [40, p. 15].

$$
\mathrm{D}_{q,z}F(z,\alpha,q)=F(z,\alpha+1,q). \tag{93}
$$

We will use a slightly modified approach of [39] to reestablish formulas in Jackson [23].

Definition 8. Let $\{\theta_i\}_0^\infty$ and $\{\phi_i\}_0^\infty$ denote arbitrary secuences. The Carlitz–Gould q-difference operator is defined by

$$
\triangle_{\mathrm{CG},q}\theta_0\equiv\theta_1-\theta_0,\ \triangle_{\mathrm{CG},q}^{n+1}\theta_0\equiv\triangle_{\mathrm{CG},q}^{n}\theta_1-q^n\triangle_{\mathrm{CG},q}^{n}\theta_0. \tag{94}
$$

This implies

$$\triangle_{\mathrm{CG},q}^{n}\theta_0 = (E \boxminus_q 1)^n = \prod_{l=0}^{n-1}(E-q^l)\theta_0 = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}_q q^{\binom{n-k}{2}}\theta_k. \tag{95}$$

$$\theta_n = \sum_{k=0}^{n}\binom{n}{k}_q \triangle_{\mathrm{CG}q}^{k}\theta_0. \tag{96}$$

$$\triangle_{\mathrm{CG},q}^{n}(\theta_0\phi_0) = \sum_{k=0}^{n}\binom{n}{k}_q \triangle_{\mathrm{CG},q}^{k}\theta_0 \triangle_{\mathrm{CG},q}^{n-k}\phi_k. \tag{97}$$

Theorem 6.2. [22, (2), p.145], [23, (7), p.146].

$$\sum_{k=1}^{\infty} x^k\theta_k = \sum_{k=1}^{\infty}\frac{x^k}{(x;q)_k}\triangle_{\mathrm{CG},q}^{k-1}\theta_1 \tag{98}$$

Proof.

$$\begin{aligned}
&RHS = \sum_{k=1}^{\infty} x^k \sum_{l=0}^{\infty} x^l \binom{k+l-1}{l}_q \triangle_{\mathrm{CG},q}^{k-1}\theta_1 = \\
&\sum_{m=0}^{\infty} x^{m+1}\sum_{k=0}^{m}\binom{m}{k}_q \triangle_{\mathrm{CG},q}^{k}\theta_1 = \sum_{m=0}^{\infty} x^{m+1}\sum_{k=0}^{m}\binom{m}{k}_q \prod_{l=0}^{k-1}(E-q^l)\theta_1 \\
&= \sum_{m=0}^{\infty} x^{m+1}E^m\theta_1 = LHS.
\end{aligned} \tag{99}$$

□

Theorem 6.3. [23, (8), p.146].

$$\sum_{k=0}^{\infty}\frac{x^k\theta_k}{\{k\}_q!} = \mathrm{E}_q(x)\sum_{k=0}^{\infty}\frac{x^k\triangle_{\mathrm{CG},q}^{k}}{\{k\}_q!}\theta_0. \tag{100}$$

Proof. Put $s_q \equiv \sum_{s=0}^{\infty}\frac{x^k\triangle_{\mathrm{CG},q}^{k}}{\{k\}_q!}\theta_0$. Then

$$s_q = \sum_{s=0}^{\infty} x^s \sum_{r=0}^{s}\frac{(-1)^{r+s}q^{\binom{s-r}{2}}}{\{r\}_q!\{s-r\}_q!}\theta_r = E_{\frac{1}{q}}(-x)\sum_{k=0}^{\infty}\frac{x^k\theta_k}{\{k\}_q!}. \tag{101}$$

□

Theorem 6.4. [23, (9), p.146]. *Let* $\Phi(x) \equiv \sum_{k=0}^{\infty} x^k\phi_k$. *Then*

$$\sum_{k=0}^{\infty} x^k\theta_k\phi_k = \sum_{k=0}^{\infty}\frac{\mathrm{D}_q^k\Phi(x)\triangle_{\mathrm{CG},q}^{k}}{\{k\}_q!}\theta_0. \tag{102}$$

Proof.

$$LHS = \sum_{k=0}^{\infty} x^k \phi_k \sum_{n=0}^{k} \binom{k}{n}_q \triangle_{\mathrm{CG}q}^n \theta_0 = \sum_{k=0}^{\infty} \frac{x^k}{\{k\}_q!} \triangle_{\mathrm{CG}q}^k \theta_0 \mathrm{D}_q^k \left(\sum_{l=0}^{\infty} x^l \phi_l \right). \tag{103}$$

□

Theorem 6.5. *Almost a q-analogue of* [40, (14), p. 62]. *The unique solution of Truesdells F_q-equation (93), such that $F(0, \alpha, q) = \Phi(\alpha, q)$ is given by*

$$F(z, \alpha, q) = \mathrm{E}_q(z) \sum_{k=0}^{\infty} \frac{z^k \triangle_{\mathrm{CG},q}^k}{\{k\}_q!} \Phi(\alpha, q), \tag{104}$$

where $\triangle_{\mathrm{CG},q}$ refers to α.

Proof. We see that the boundary condition is fullfilled. The product rule for D_q gives

$$\mathrm{D}_q F(z, \alpha, q) = \mathrm{E}_q(z) \left(\sum_{k=0}^{\infty} \frac{z^{k-1} \triangle_{\mathrm{CG},q}^{k+1}}{\{k-1\}_q!} + \sum_{k=0}^{\infty} \frac{(zq)^k \triangle_{\mathrm{CG},q}^k}{\{k\}_q!} \right) \Phi(\alpha, q) = \mathrm{E}_q(z) \sum_{k=0}^{\infty} \frac{z^k \triangle_{\mathrm{CG},q}^k}{\{k\}_q!} \Phi(\alpha + 1, q) = F(z, \alpha + 1, q). \tag{105}$$

□

7. Appendix: q-Integral theorems

In this appendix, we show how the most important integral theorems are connected with q-analysis.

The q-integral

$$\int_0^a f(t, q)\, d_q(t) \equiv a(1-q) \sum_{n=0}^{\infty} f(aq^n, q) q^n \tag{106}$$

can be written in der form

$$\int f d\mu = \sum_{n=0}^{\infty} b_n \mu(E_n), \tag{107}$$

where b_n means the function value $f(aq^n, q)$ times a and $\mu(E_n) = (1-q)q^n$ denotes the measure in the point $x = aq^n$. We use a σ-Algebra

$\mathcal{M} = \{\mu(E_n) = (1-q)q^n\}_{n=0}^{\infty}$, where the sets $\{E_n\}_0^\infty$ are disjoint. By the definition of measures we have

$$\mu(\cup_0^\infty E_n) = \sum_{n=0}^{\infty} \mu(E_n). \tag{108}$$

According to Jackson we put

$$\int_0^\infty f(t,q)\, d_q(t) \equiv (1-q) \sum_{n=-\infty}^{\infty} f(q^n,q)q^n,\ 0<|q|<1, \tag{109}$$

provided that the sum converges absolutely. Similarly, we have for the q-integral (109):

$$\int f d\mu = \sum_{n=-\infty}^{\infty} b_n \mu(E_n), \tag{110}$$

where b_n denotes the function value $f(q^n,q)$ and $\mu(E_n) = (1-q)q^n$ denotes the measure in the point $x = q^n$.

The following three theorems are proved analogously to the standard case.

Theorem 7.1. *Triangle inequality*

$$|\int f d\mu| \leq \int |f| d\mu. \tag{111}$$

Theorem 7.2. *Let* $\{f_n\}_0^\infty$ *be a continuous sequence with limit function*

$$f = \lim_{n\to\infty} f_n, \tag{112}$$

which converges uniformly. Then we have

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu. \tag{113}$$

Proof.

$$\begin{aligned} &|\int f_n d\mu - \int f d\mu| \leq \int |fn - f| d\mu \leq \\ &\leq \int ||fn-f|| d\mu = ||fn-f|| \int d\mu, \to 0, n \to \infty. \end{aligned} \tag{114}$$

□

Theorem 7.3. *If the conditions in (112) are fullfilled, we have*

$$\int \sum_{n=0}^{\infty} f_n(x) d\mu = \sum_{n=0}^{\infty} \int f_n(x) d\mu \tag{115}$$

Proof. Use (113). □

8. Conclusion

We have considered the Γ_q function as a function of a complex variable z (and complex q). The main results are the balanced Γ_q functions. The formula (85) could also be written with tilde instead of $+b$, just as for the q-factorial. To be able to treat a general unit root of unity, we introduce the generalized tilde operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

as follows:

$$a \mapsto a + \frac{2\pi i m}{l \log q}, \ (m,l) = 1. \tag{116}$$

This means

$$\langle \widetilde{\tfrac{m}{l}a}; q\rangle_n \equiv \prod_{m=0}^{n-1}(1 - e^{2\pi i \frac{m}{l}} q^{a+m}). \tag{117}$$

We can therefore insert a generalized tilde operator $\widetilde{\tfrac{m}{l}}$ in a balanced Γ_q function. This is equivalent to an addition with the corresponding complex number. We have not included the $q-$ Stirling formula [5, S. 899] in this report, because it fits better in connection with the Appell functions [8]. An article on this subject is being prepared.

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