ON q-REAL NUMBERS

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ABSTRACT. We introduce the q-real numbers $\mathbb{R}_q$ as q-additions of real numbers of absolute value less than one. We have a one-to-one correspondence between the convergence regions of the two q–Lauricella functions $\Phi_{\mathrm{A}}^{(n)}$ and $\Phi_{\mathrm{C}}^{(n)}$ and the existence of q-real numbers with n letters. In the same way, there is a one-to-one correspondence between the convergence regions of the same multiple functions with one factor $q^{\binom{k}{2}}$ and the same q-real numbers, but with one corresponding JHC q-addition. The q-dependent norm of a q-real number is defined as the maximum value of the corresponding q-addition. **keywords** : NOVA q-addition; q-real number; convergence region; numerical values;

1. Introduction

The general definition of q-real numbers has been given in [4, p. 98], they are the real numbers generated by the two q-additions. The purpose of the q-real numbers $\mathbb{R}_q$ is to give a convenient notation for q-additions in formal power series, in particular for q-exponential and q-trigonometric functions. The purpose of this paper is to discuss convergence of powers of three real letters with length less than one. It will take too long to do computations with more than three letters, but the tendency is clear.

2. Numerical values for q-real numbers

The following functions will be our main concern in this paper:

Definition 1. The function $\mathrm{F}(n)_{a,b,q}$ is defined by

$$\mathrm{F}(n)_{a,b,q} : n \mapsto (a \oplus_q b)^n, 0 < a < 1, 0 < b < 1, 0 < q < 1. \tag{1}$$

The function $\mathrm{f}(n)_{a,b,q}$ is defined by

$$\mathrm{f}(n)_{a,b,q} : n \mapsto (b \boxplus_q a)^n, 0 < a < 1, 0 < b < 1, 0 < q < 1. \tag{2}$$

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In the papers [3] and [8], we concluded that the functions F(n) and G(n) (with three additions instead of two) posess a unique, q-dependent maximum. *Mathematica* computations show that also f(n) has this property.

The worst conditions for convergence are for $\mathrm{F}(n)_{a,b,q}$ when a and b are close to 1. We illustrate some of these cases: The following table (I) shows some function values $\mathrm{F}(n)_{.995,.995,.895}$:

n	$\mathrm{F}(n)_{.995,.995,.895}$	n	$\mathrm{F}(n)_{.995,.995,.895}$
410	1.6812×10^7	430	1.60521×10^7
450	1.52843×10^7	470	1.45169×10^7
490	1.37568×10^7	510	1.30096×10^7
530	1.22798×10^7	550	1.15709×10^7
570	1.08854×10^7	590	1.02255×10^7
610	9.59238×10^6		

The function $\mathrm{F}(n)_{.995,.995,.895}$ has its maximum 2.08494×10^7 for $n = 249$.

The function $\mathrm{F}(n)_{.996,.996,.895}$ has its maximum 2.74055×10^7 for $n = 299$.

Finally we show some function values for $\mathrm{F}(n)_{.999,.999,.895}$:

n	$\mathrm{F}(n)_{.999,.999,.895}$	n	$\mathrm{F}(n)_{.999,.999,.895}$
760	1.20952×10^8	800	1.2275×10^8
900	1.25863×10^8	1000	1.2727×10^8
1200	1.26114×10^8		

The following tables show the monotonic behaviour of the function $\mathrm{f}(n)_{a,b,.999}$, for short f(n):

n	$\mathrm{f}(n)_{.45,.55,.99}$	n	$\mathrm{f}(n)_{.45,.55,.99}$
9	.851913	25	.273621
50	.00644712	60	.000786607

n	$\mathrm{f}(n)_{.54,.54,.88}$	n	$\mathrm{f}(n)_{.54,.54,.88}$
2	1.09642	3	1.05056
10	.213432	20	.00208828

n	$\mathrm{f}(n)_{.6,.6,.88}$	n	$\mathrm{f}(n)_{.6,.6,.88}$
2	1.3536	4	1.4539
10	.612116	20	.0171767

The function $\mathrm{f}(n)_{.6,.6,.88}$ has maximum for $n = 4$.

n	$\mathrm{f}(n)_{.8,.8,.88}$	n	$\mathrm{f}(n)_{.8,.8,.88}$
2	2.4064	4	4.59504
10	10.8698	20	5.41644
30	0.916224	40	0.111917

The function $\mathrm{f}(n)_{.8,.8,.88}$ has its maximum 11.1176 for $n = 11$.

n	$\mathrm{f}(n)_{.9,.9,.88}$	n	$\mathrm{f}(n)_{.9,.9,.88}$
2	3.0456	4	7.36036
10	35.2977	20	57.1168
30	31.3745	40	12.4451
50	4.49883	60	1.58452
70	.554039	80	.193333

The function $\mathrm{f}(n)_{.9,.9,.88}$ has its maximum 58.9032 for $n = 18$.

n	$\mathrm{f}(n)_{.95,.95,.88}$	n	$\mathrm{f}(n)_{.95,.95,.88}$
2	3.3934	4	9.13742
10	60.6118	20	168.417
30	158.858	40	108.204
60	40.6222	80	14.6148

The function $\mathrm{f}(n)_{.95,.95,.88}$ has its maximum 176.238 for $n = 24$.

For the following definition one could compare with the formula from [4, p.110]:

$$(\boxplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \boxplus_q a_1 x \boxplus_q \ldots)^k \equiv \sum_{|m|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q q^{\binom{\vec{n}}{2}}, \quad (3)$$

where $\vec{n} = (m_2, \ldots)$.

Definition 2. If a_q is the q-real number with j letters $\oplus_{q,l=0}^{j-1} a_l$, $|a_l| < 1,\ \forall l$, its k'th power is given by

$$(\oplus_{q,l=0}^{j-1} a_l)^k \equiv (a_0 \oplus_q a_1 \oplus_q \ldots)^k \equiv \sum_{|m|=k} \prod_{l=0}^{j-1} (a_l)^{m_l} \binom{k}{\vec{m}}_q , \quad (4)$$

where for each JHC-addition in a_i, we multiply by $q^{\binom{m_i}{2}}$.

Definition 3. The norm $M(q)$ or $|a_q|$ of the q-real number a_q, with j letters, is defined as the absolute maximum of the k'th power given by (4).

If there is no absolute maximum, and the function in (4) is infinite, we say that there is no q-real number a_q.

We will give one example of a q-real number, where the norm in (4) is infinite in formula (13). We make a list of function values $M(q)$.

We begin with $q = .86$.

a	b	$M(.86)(a \oplus_q b)$	a	b	$M(.86)(a \oplus_q b)$
.35	.39	.74	.39.	.39	.78
.41	.39	.8	.41.	.44	.85
.49	.44	.93	.49.	.495	.985
.51	.495	1.005	.51	.51	1.02
.51	.55	1.08433	.53	.55	1.13366
.55	.55	1.19777	.55	.58	1.32432
.59	.58	1.55709	.62	.58	1.80075
.62	.617	2.20245	.62	.65	2.70001
.65	.65	3.30482	.65	.69	4.44303
.695	.69	6.4194	.72	.69	8.01034
.72	.73	11.6574	.72	.74	12.855
.72	.76	15.8092	.74	.76	19.4748
.82	.76	49.4521	.82	.79	71.6071
.82	.81	92.8952	.82	.83	122.088
.85	.83	187.617	.85	.86	294.241
.88	.86	476.45	.88	.89	790.773
.91	.89	1366.28	.91	.92	2441.46
.94	.92	4634.56	.94	.95	9295.66

(5)

a	b	$M(.86)(b \boxplus_q a)$	a	b	$M(.86)(b \boxplus_q a)$
.35	.39	.74	.39.	.39	.78
.41	.39	.8	.41..	.44	.85
.49	.44	.93	.49.	.495	.985
.51	.495	1.005	.51	.51	1.02
.51	.55	1.06	.53	.55	1.08626
.55	.55	1.1253	.55	.58	1.18989
.59	.58	1.29308	.62	.58	1.38734
.62	.617	1.54767	.62	.65	1.73967
.65	.65	1.90042	.65	.69	2.26059
.695	.69	6.4194	.72	.69	8.01034
.72	.73	11.6574	.72	.74	12.855
.72	.76	15.8092	.74	.76	19.4748
.82	.76	49.4521	.82	.79	71.6071
.82	.81	92.8952	.82	.83	122.088
.85	.83	12.7757	.85	.86	16.9608
.88	.86	19.4189	.88	.89	26.8724
.91	.89	30.7857	.91	.92	44.8144
.94	.92	51.5819	.94	.95	80.4265

(6)

For the following tables, compare with [3].

a	b	c	$M(.86)(a \oplus_q b \oplus_q c)$
.35	.39	.44	1.46716
.35	.39	.49	1.78452
.39	.39	.49	2.12849
.41	.39	.49	2.33556
.41	.39	.52	2.73846
.41	.39	.55	3.2365
.41	.44	.55	4.31919
.49	.44	.55	7.2276
.49	.495	.55	10.5986
.49	.495	.595	15.0459
.51	.495	.595	17.5385
.51	.495	.625	22.5684
.51	.51	.625	25.4686
.51	.55	.625	35.4274
.53	.55	.625	42.0194
.55	.55	.625	49.8303
.55	.58	.625	65.1742
.59	.58	.625	93.7617
.59	.58	.645	113.775
.59	.58	.665	139.302
.62	.58	.665	186.202

(7)

a	b	c	$M(.86)(a \oplus_q b \oplus_q c)$
.62	.58	.685	230.106
.62	.617	.685	331.985
.62	.617	.7	390.658
.62	.65	.75	551.787
.65	.65	.7	757.642
.65	.69	.7	1179.81
.65	.69	.72	1488.97
.65	.69	.74	1897.32
.695	.69	.74	3165.52
.72	.69	.74	4277.74
.72	.73	.74	6944.15
.72	.73	.77	10256.7
.72	.74	.77	11661.5
.72	.76	.77	15119.1
.74	.76	.77	19534.2
.82	.76	.77	59470.2
.82	.76	.8	91037.
.82	.79	.8	138902.
.82	.81	.8	186524.
.82	.81	.84	345902.
.82	.83	.8	253762.
.82	.83	.84	472049.
.85	.83	.84	766211.
.85	.86	.84	1.26191×10^6
.85	.86	.87	2.11131×10^6
.88	.86	.87	3.59495×10^6
.88	.89	.87	6.23899×10^6
.88	.89	.9	1.10795×10^7
.91	.89	.9	2.01627×10^7
.91	.92	.9	3.77592×10^7
.91	.92	.93	7.31871×10^7
.94	.92	.93	1.47806×10^8
.94	.95	.93	3.14342×10^8

(8)

a	b	c	$M(.86)(b \oplus_q c \boxplus_q a)$
.35	.39	.44	1.37428
.35	.39	.49	1.62386
.39	.39	.49	1.84463
.41	.39	.49	1.9747
.41	.39	.52	2.23517
.41	.39	.55	2.58912
.41	.44	.55	3.31897
.49	.44	.55	4.61005
.49	.495	.55	6.38747
.49	.495	.595	8.62386
.51	.495	.595	9.46487
.51	.495	.625	11.7292
.51	.51	.625	13.0834
.51	.55	.625	17.5385
.53	.55	.625	19.3498
.55	.55	.625	21.326
.55	.58	.625	26.9449
.59	.58	.625	32.8342
.59	.58	.645	38.9074
.59	.58	.665	46.5103
.62	.58	.665	53.9569

(9)

a	b	c	$M(.86)(b \oplus_q c \boxplus_q a)$
.62	.58	.685	64.7229
.62	.617	.685	90.0825
.62	.617	.7	104.528
.62	.65	.75	142.84
.65	.65	.7	166.557
.65	.69	.7	249.47
.65	.69	.72	308.313
.65	.69	.74	385.677
.695	.69	.74	485.145
.72	.69	.74	550.087
.72	.73	.74	860.186
.72	.73	.77	1232.07
.72	.74	.77	1387.7
.72	.76	.77	1768.49
.74	.76	.77	1960.15
.82	.76	.77	2935.43
.82	.76	.8	4328.6
.82	.79	.8	6398.17
.82	.81	.8	8409.91
.82	.81	.84	14952.9
.82	.83	.8	11194.2
.82	.83	.84	20033.1
.85	.83	.84	23290.8
.85	.86	.84	37153.7
.85	.86	.87	60299.1
.88	.86	.87	70084.5
.88	.89	.87	117358.
.88	.89	.9	201456.
.91	.89	.9	233976.
.91	.92	.9	419615.
.91	.92	.93	779802.
.94	.92	.93	904753.
.94	.95	.93	1.8057×10^6

(10)

Next we consider the case $q = .96$.

a	b	$M(.96)(a \oplus_q b)$	a	b	$M(.96)(a \oplus_q b)$
.35	.39	.74	.39.	.39	.78
.41	.39	.8	.41.	.44	.85
.49	.44	.93	.49.	.495	.985
.51	.495	1.005	.51	.51	1.03
.51	.55	1.1892	.53	.55	1.40523
.55	.55	1.67081	.55	.58	2.30528
.59	.58	3.94747	.62	.58	6.41433
.62	.617	12.6778	.62	.65	25.3753
.65	.65	50.5243	.65	.69	140.279
.695	.69	493.273	.72	.69	1056.05
.72	.73	3848.31	.72	.74	5419.5
.72	.76	11022.	.74	.76	22509.3
.82	.76	558298.	.82	.79	1.97549×10^6
.82	.81	4.81429×10^6	.82	.83	1.22486×10^7
.85	.83	5.3152×10^7	.85	.86	2.45543×10^8
.88	.86	1.25202×10^9	.88	.89	6.86886×10^9
.91	.89	4.27002×10^{10}	.91	.92	2.91042×10^{11}
.94	.92	2.36258×10^{12}	.94	.95	2.19089×10^{13}

(11)

a	b	$M(.96)(b \boxplus_q a)$	a	b	$M(.96)(b \boxplus_q a)$
.35	.39	.74	.39.	.39	.78
.41	.39	.8	.41.	.44	.85
.49	.44	.93	.49.	.495	.985
.51	.495	1.005	.51	.51	1.02
.51	.55	1.1254	.53	.55	1.21011
.55	.55	1.32139	.55	.58	1.57956
.94	.92	899546.	.94	.95	4.54453×10^6

(12)

To determine the convergence region, it will be sufficient to write the values of $M(q)$ together with parameter values as in the following table:

q	a	b	c	$M(q)(a \oplus_q b \oplus_q c)$
.96	.94	.95	.93	3.37315×10^{28}
.97	.94	.95	.93	7.69869×10^{37}
.98	.94	.95	.93	4.03485×10^{56}
.99	.94	.95	.93	∞

(13)

Finally, we have $M(.99)(.93 \oplus_q .95 \boxplus_q .94) \approx 1.77812 \times 10^{85}$.

We have shown that the q-real numbers have the expected properties, i.e. q-dependent convergence regions with larger convergence regions with more JHC-additions. We have found a unique norm for each q-real number.

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