SOME RESULTS FOR q-LAGUERRE POLYNOMIALS.

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Contents

1. Introduction

The purpose of this paper is to continue the study of q-special functions by the method outlined in [14], [15] and [16].

We will use the generating function technique by Rainville [26] to prove recurrences for q-Laguerre polynomials, which are q-analogues of results in [26]. Some of these recurrences were stated already by Moak [24].

We will also find q-analogues of Carlitz' [7] operator expression for Laguerre polynomials. The notation for Cigler's [13] operational calculus will be used when needed. As an application, q-analogues of bilinear generating formulas for Laguerre polynomials of Chatterjea [12, p.57], [11, p.88] will be found.

We begin with a few definitions.

Definition 1. The power function is defined by $q^a = e^{alog(q)}$. We always use the principal branch of the logarithm.

The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q = \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{1}$$

$$\{n\}_q! = \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q! = 1,\ q \in \mathbb{C}, \tag{2}$$

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Definition 2. The q-hypergeometric series was developed by Heine 1846 as a generalization of the hypergeometric series:

$$ {}_2\phi_1(a,b;c|q,z) = \sum_{n=0}^{\infty} \frac{\langle a;q\rangle_n \langle b;q\rangle_n}{\langle 1;q\rangle_n \langle c;q\rangle_n} z^n, \tag{3} $$

with the notation for the q-shifted factorial (compare [21, p.38])

$$ \langle a;q\rangle_n = \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots, \end{cases} \tag{4} $$

which is introduced in this paper.

Remark 1. The relation to Watson's notation, which is also included in the method, is

$$ \langle a;q\rangle_n = (q^a;q)_n, \tag{5} $$

where

$$ (a;q)_n = \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\ldots \end{cases} \tag{6} $$

Definition 3. Furthermore,

$$ (a;q)_\infty = \prod_{m=0}^{\infty}(1-aq^m),\ 0<|q|<1. \tag{7} $$

$$ (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ a \neq q^{-m-\alpha}, m=0,1,\ldots. \tag{8} $$

Definition 4. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod$\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R}\times e^{2\pi i\theta},\ \theta\in\mathbb{R}$. The operator

$$ \widetilde{\ }: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}} $$

is defined by

$$ a \mapsto a + \frac{\pi i}{\log q}. \tag{9} $$

Furthermore we define

$$ \widetilde{\langle a;q\rangle_n} \equiv \langle \widetilde{a};q\rangle_n. \tag{10} $$

By (9) it follows that

$$\widetilde{\langle a;q\rangle_n}=\prod_{m=0}^{n-1}(1+q^{a+m}), \tag{11}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$.

The following simple rules follow from (9).

$$\widetilde{a}\pm b=\widetilde{a\pm b}, \tag{12}$$

$$\widetilde{a}\pm\widetilde{b}=a\pm b, \tag{13}$$

$$q^{\widetilde{a}}=-q^a, \tag{14}$$

where the second equation is a consequence of the fact that we work $\mathrm{mod}\frac{2\pi i}{\log q}$.

Definition 5. Generalizing Heine's series, we shall define a q-hypergeometric series by (compare [20, p.4]):

$$\begin{aligned}&{}_p\phi_r(\hat{a}_1,\ldots,\hat{a}_p;\hat{b}_1,\ldots,\hat{b}_r|q,z)\equiv\ {}_p\phi_r\left[\begin{array}{c}\hat{a}_1,\ldots,\hat{a}_p\\ \hat{b}_1,\ldots,\hat{b}_r\end{array}|q,z\right]=\\ &=\sum_{n=0}^{\infty}\frac{\langle\hat{a}_1,\ldots,\hat{a}_p;q\rangle_n}{\langle 1,\hat{b}_1,\ldots,\hat{b}_r;q\rangle_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+r-p}z^n,\end{aligned} \tag{15}$$

where $q\neq 0$ when $p>r+1$, and

$$\widehat{a}=\begin{cases}a\\ \widetilde{a}\end{cases} \tag{16}$$

We will skip the $\hat{a}$ for the rest of the paper.

Definition 6. The following generalization of (15) will sometimes be used:

$$\begin{aligned}&{}_{p+p'}\phi_{r+r'}(a_1,\ldots,a_p;b_1,\ldots,b_r|q,z||s_1,\ldots,s_{p'};t_1,\ldots,t_{r'})\equiv\\ &{}_{p+p'}\phi_{r+r'}\left[\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_r\end{array}|q,z||\begin{array}{c}s_1,\ldots,s_{p'}\\ t_1,\ldots,t_{r'}\end{array}\right]=\\ &=\sum_{n=0}^{\infty}\frac{\langle a_1;q\rangle_n\ldots\langle a_p;q\rangle_n}{\langle 1;q\rangle_n\langle b_1;q\rangle_n\ldots\langle b_r;q\rangle_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+r+r'-p-p'}\times\\ &z^n\prod_{k=1}^{p'}(s_k;q)_n\prod_{k=1}^{r'}(t_k;q)_n^{-1},\end{aligned} \tag{17}$$

where $q\neq 0$ when $p+p'>r+r'+1$.

Remark 2. Equation (17) is used in certain special cases when we need factors $(t;q)_n$ in the q-series.

Definition 7. Let the q-Pochhammer symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} = \prod_{m=0}^{n-1} \{a+m\}_q. \tag{18}$$

An equivalent symbol is defined in [17, p.18] and is used throughout that book. See also [2, p.138].

This quantity can be very useful in some cases where we are looking for q-analogues and it is included in the new notation.

Definition 8. With the help of the q-gamma function

$$\Gamma_q(x) = \frac{\langle 1;q\rangle_\infty}{\langle x;q\rangle_\infty}(1-q)^{1-x},\ 0<q<1, \tag{19}$$

we can define the two Jackson q-Bessel functions

$$J_\alpha^{(1)}(z;q) = \frac{\langle \alpha+1;q\rangle_\infty}{\langle 1;q\rangle_\infty}\left(\frac{z}{2}\right)^\alpha {}_2\phi_1\left(\infty,\infty;\alpha+1|q,-\frac{z^2}{4}\right), \tag{20}$$

$$J_\alpha^{(2)}(z;q) = \frac{\langle \alpha+1;q\rangle_\infty}{\langle 1;q\rangle_\infty}\left(\frac{z}{2}\right)^\alpha {}_0\phi_1\left(-;\alpha+1|q,-\frac{z^2 q^{\alpha+1}}{4}\right). \tag{21}$$

Definition 9. The Euler-Jackson q-difference operator is given by

$$\left(D_q\varphi\right)(x) = \frac{\varphi(x)-\varphi(qx)}{(1-q)x},\ q\in\mathbb{C}\backslash\{1\}. \tag{22}$$

The limit as q approaches 1 is the derivative

$$\lim_{q\to 1}\left(D_q\varphi\right)(x) = \frac{d\varphi}{dx}, \tag{23}$$

if φ is differentiable at x.

If we want to indicate the variable which the q-difference operator is applied to, we denote the operator $\left(D_{q,x}\varphi\right)(x,y)$.

We will use a notation introduced by Burchnall and Chaundy.

$$\theta_1 \equiv xD_{q,x},\ \theta_2 \equiv yD_{q,y}. \tag{24}$$

Definition 10. If $|q|>1$, or
$0<|q|<1$ and $|z|<|1-q|^{-1}$, the q-exponential function $E_q(z)$ was defined by Jackson 1904.

$$E_q(z) = \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{25}$$

For $0 < |q| < 1$ we can define $E_q(z)$ for all other values of z by analytic continauation.

The q-difference equation for $E_q(z)$ is

$$D_q E_q(az) = aE_q(az). \tag{26}$$

Two q-analogues of the trigonometric functions are defined by

$$Sin_q(x) = \frac{1}{2i}(E_q(ix) - E_q(-ix)), \tag{27}$$

and

$$Cos_q(x) = \frac{1}{2}(E_q(ix) + E_q(-ix)). \tag{28}$$

2. Generating functions and recurrences for q-Laguerre polynomials

In this paper we will be working with two different q-Laguerre polynomials. The polynomial $L^{(\alpha)}_{n,q,c}(x)$ was used by Cigler [13].

$$\begin{aligned} L^{(\alpha)}_{n,q,c}(x) &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}_q \frac{\{n\}_q!}{\{k\}_q!} q^{k^2+\alpha k}(-1)^k x^k \\ &\equiv \sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2+k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n} \\ &\equiv \frac{\langle \alpha+1;q\rangle_n}{(1-q)^n} {}_1\phi_1\left(-n;\alpha+1|q,-x(1-q)q^{n+\alpha+1}\right). \end{aligned} \tag{29}$$

The most common q-Laguerre polynomial $L^{(\alpha)}_{n,q}(x)$ is defined as follows. Except for the notation, this definition is equivalent to [24], [20] and [28].

$$L^{(\alpha)}_{n,q}(x) = \frac{L^{(\alpha)}_{n,q,c}(x)}{\{n\}_q!} \tag{30}$$

In [22] the q-Laguerre polynomial is defined as

$$\frac{\langle \alpha+1;q\rangle_n}{\langle 1;q\rangle_n} {}_1\phi_1\left(-n;\alpha+1|q,-xq^{n+\alpha+1}\right). \tag{31}$$

Consider sets $\sigma_n(x)$ defined by

$$E_q(t)\Psi(xt) = \sum_{n=0}^{\infty} \sigma_n(x)t^n. \tag{32}$$

Let

$$F = E_q(t)\Psi(xt). \tag{33}$$

Then

$$D_{q,x}F = tE_q(t)D_q\Psi, \tag{34}$$

$$D_{q,t}F = E_q(t)\Psi + x(1-(1-q)t)E_q(t)D_q\Psi. \tag{35}$$

An elimination of Ψ and $D_q\Psi$ from the above equations gives

$$x(1-(1-q)t)D_{q,x}F - tD_{q,t}F = -tF, \tag{36}$$

and

$$\sum_{n=0}^{\infty} xD_q\sigma_n(x)t^n - \sum_{n=1}^{\infty} x(1-q)D_q\sigma_{n-1}(x)t^n - \sum_{n=0}^{\infty}\{n\}_q\sigma_n(x)t^n =$$

$$= -\sum_{n=1}^{\infty}\sigma_{n-1}(x)t^n.$$

By equating the coefficients of t^n we obtain the following recurrence:

$$D_q\sigma_0(x) = 0. \tag{37}$$

$$xD_q\sigma_n(x) - x(1-q)D_q\sigma_{n-1}(x) - \{n\}_q\sigma_n(x) = -\sigma_{n-1}(x),\ n \geq 1. \tag{38}$$

In particular, by (53) we obtain the following recurrence for the q-Laguerre polynomials, which is a q-analogue of [26, p.134]:

$$\begin{aligned} &xD_qL_{n,q}^{(\alpha)}(x) - x(1-q)\{\alpha+n\}_qD_qL_{n-1,q}^{(\alpha)}(x) = \\ &\{n\}_qL_{n,q}^{(\alpha)}(x) - \{\alpha+n\}_qL_{n-1,q}^{(\alpha)}(x). \end{aligned} \tag{39}$$

Now let's assume that Ψ has the formal power series expansion

$$\Psi(u) = \sum_{n=0}^{\infty}\gamma_n u^n. \tag{40}$$

Then

$$\sum_{n=0}^{\infty}\sigma_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\gamma_k x^k t^n}{\{n-k\}_q!}, \tag{41}$$

so that

$$\sigma_n(x) = \sum_{k=0}^{n}\frac{\gamma_k x^k}{\{n-k\}_q!}. \tag{42}$$

Now by the q-binomial theorem

$$\begin{aligned}
&\sum_{n=0}^{\infty}\{c\}_{n,q}\sigma_n(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\{c\}_{n,q}\gamma_k x^k t^n}{\{n-k\}_q!}=\\
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\{c\}_{n+k,q}\gamma_k x^k t^{n+k}}{\{n\}_q!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\{c+k\}_{n,q}t^n}{\{n\}_q!}\frac{\{c\}_{k,q}\gamma_k(xt)^k}{1}=\\
&\sum_{k=0}^{\infty}\{c\}_{k,q}\gamma_k(xt)^k\frac{(tq^{c+k};q)_\infty}{(t;q)_\infty}=\sum_{k=0}^{\infty}\frac{\{c\}_{k,q}\gamma_k(xt)^k}{(t;q)_{c+k}}.
\end{aligned}\tag{43}$$

As a special case we get the following generating function which is a q-analogue of [18, p.43, (73)], [26, p.135, (13)].

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\{c\}_{n,q}L_{n,q}^{(\alpha)}(x)t^n}{\{1+\alpha\}_{n,q}}=\sum_{n=0}^{\infty}\frac{\{c\}_{n,q}\ q^{n^2+\alpha n}(-xt)^n}{\{n\}_q!\{1+\alpha\}_{n,q}(t;q)_{c+n}}\\
&\equiv\frac{1}{(t;q)_c}{}_1\phi_2(c;1+\alpha|q;-xtq^{1+\alpha}(1-q)||-;tq^c).
\end{aligned}\tag{44}$$

Consider the important case $c=1+\alpha$ in (44). This is equivalent to [24, p.29 4.17], [1, p.132 4.2], [19, p.120 11']. Call the RHS $F(x,t,q,\alpha)$. By computing the q-difference of $F(x,t,q,\alpha)$ with respect to x we obtain

$$D_{q,x}F=-tq^{1+\alpha}F(qx,t,q,\alpha+1).\tag{45}$$

Equating coefficients of t^n, we obtain the following recurrence relation which is a q-analogue of [26, p.203]. Also compare with [22, p.109, 3.21.8] and [23, p.79].

$$D_qL_{n,q}^{(\alpha)}(x)=-q^{1+\alpha}L_{n-1,q}^{(1+\alpha)}(xq).\tag{46}$$

By computing the q-difference of $F(x,t,q,\alpha)$ with respect to t and equating coefficients of t^n, we obtain

$$\{n+1\}_qL_{n+1,q}^{(\alpha)}(x)=\{\alpha+1\}_qL_{n,q}^{(\alpha+1)}(x)+\frac{L_{n+1,q}^{(\alpha+1)}(\frac{x}{q})-L_{n+1,q}^{(\alpha+1)}(x)}{1-q}.\tag{47}$$

Proof.

$$D_{q,t}F=\sum_{n=0}^{\infty}\frac{q^{n^2+n\alpha}(-x)^n((t;q)_{\alpha+1+n}\{n\}_q t^{n-1}-t^nD_q(t;q)_{\alpha+1+n})}{(tq;q)_{\alpha+1+n}(t;q)_{\alpha+1+n}\{n\}_q!}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2+n\alpha}(-x)^nt^n((t;q)_{\alpha+1+n}\{n\}_q t^{-1}+\{\alpha+1+n\}_q(tq;q)_{\alpha+n})}{(tq;q)_{\alpha+1+n}(t;q)_{\alpha+1+n}\{n\}_q!}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2+n\alpha}(-x)^nt^n(\{n\}_q\frac{1-t}{t}+\{\alpha+1+n\}_q)}{(t;q)_{\alpha+2+n}\{n\}_q!}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2+n\alpha+n}\{\alpha+1\}_q(-xt)^n}{(t;q)_{\alpha+2+n}\{n\}_q!}+\sum_{n=0}^{\infty}\frac{q^{n^2+n\alpha}\{n\}_q(-x)^nt^{n-1}}{(t;q)_{\alpha+2+n}\{n\}_q!}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2+n\alpha+n}\{\alpha+1\}_q(-xt)^n}{(t;q)_{\alpha+2+n}\{n\}_q!}+\frac{1}{t(1-q)}\sum_{n=0}^{\infty}\frac{q^{n^2+n\alpha+n}(-xt)^n(\frac{1}{q^n}-1)}{(t;q)_{\alpha+2+n}\{n\}_q!}$$
$$=\sum_{n=0}^{\infty}t^n\{\alpha+1\}_qL_{n,q}^{(\alpha+1)}(x)+\frac{1}{1-q}\sum_{n=0}^{\infty}t^{n-1}(L_{n,q}^{(\alpha+1)}(\frac{x}{q})-L_{n,q}^{(\alpha+1)}(x))$$
$$=\sum_{n=0}^{\infty}t^n\{\alpha+1\}_qL_{n,q}^{(\alpha+1)}(x)+\frac{1}{(1-q)}\sum_{n=0}^{\infty}t^{n}(L_{n+1,q}^{(\alpha+1)}(\frac{x}{q})-L_{n+1,q}^{(\alpha+1)}(x)) \tag{48}$$

Equating coefficients of t^n we are done. □

The last equation can be expressed as

$$\{n+1\}_qL_{n+1,q}^{(\alpha)}(x)=\{\alpha+1\}_qL_{n,q}^{(\alpha+1)}(x)-xq^{2+\alpha}L_{n,q}^{(\alpha+2)}(x). \tag{49}$$

Furthermore, the relation $(1-t)F(x,t,q,\alpha+1)=F(x,tq,q,\alpha)$ yields the following mixed recurrence relation, which was already stated in [24, p.29 4.12]:

$$L_{n,q}^{(\alpha+1)}(x)-L_{n-1,q}^{(\alpha+1)}(x)=q^nL_{n,q}^{(\alpha)}(x). \tag{50}$$

By the q-binomial theorem we obtain the following equation, which is a generalization of [24, p.29 4.10] and which is a q-analogue of [26, p.209], [1, p.131 3.16], [19, p.130 38].

$$L_{n,q}^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{\langle\alpha-\beta;q\rangle_k}{\langle 1;q\rangle_k}L_{n-k,q}^{(\beta)}(x)q^{(\alpha-\beta)(n-k)},\ \alpha,\ \beta\in\mathbb{C}. \tag{51}$$

Proof.

$$\sum_{n=0}^{\infty}L_{n,q}^{(\alpha)}(x)t^n=\sum_{n=0}^{\infty}\frac{q^{n^2+\alpha n}(-xt)^n}{\{n\}_q!(t;q)_{1+\alpha+n}}=\frac{1}{(t;q)_{\alpha-\beta}}\sum_{n=0}^{\infty}\frac{q^{n^2+\beta n}(-xt)^nq^{(\alpha-\beta)n}}{\{n\}_q!(tq^{\alpha-\beta};q)_{1+\beta+n}}$$

$$= \sum_{k=0}^{\infty} \frac{\langle \alpha - \beta; q\rangle_k}{\langle 1; q\rangle_k} t^k \sum_{l=0}^{\infty} L_{l,q}^{(\beta)}(x) t^l q^{(\alpha-\beta)l}$$

Equating coefficients of t^n we are done. □

By (50) and (46) the following important recurrence obtains:

$$D_q(L_{n,q}^{(\alpha)}(x) - L_{n-1,q}^{(\alpha)}(x)) = -q^{n+\alpha} L_{n-1,q}^{(\alpha)}(xq). \tag{52}$$

The following generating function can also be found in [22, p. 109, 3.21.13]. It is a q-analogue of [18, p.43, (73")], [26, p.130], [19, p.121 12'].

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{L_{n,q}^{(\alpha)}(x) t^n}{\{1+\alpha\}_{n,q}} = E_q(t)_0\phi_1(-; 1+\alpha|q, q^{1+\alpha}(1-q)^2(-xt)) = \\ &\Gamma_q(1+\alpha)(xt)^{-\frac{\alpha}{2}} E_q(t) J_{\alpha}^{(2)}(2(1-q)\sqrt{xt}; q).\end{aligned} \tag{53}$$

Proof. Let $c \to \infty$ in (44). □

Remark 3. Another similar generating function is obtained by letting $t \to tq^{-c}$, $c \to -\infty$ in (44). These limits are q-analogues of an idea used by Feldheim [18, p.43], which is not mentioned by Rainville.

Making use of the decomposition of a series into even and odd parts from [27, p.200,208], we can rewrite (53) in the form

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{L_{2n,q}^{(\alpha)}(x) t^{2n}}{\{1+\alpha\}_{2n,q}} + \frac{t}{\{1+\alpha\}_q} \sum_{n=0}^{\infty} \frac{L_{2n+1,q}^{(\alpha)}(x) t^{2n}}{\{2+\alpha\}_{2n,q}} = E_q(t) \Bigg[{}_0\phi_7(-; \frac{1+\alpha}{2}, \\ &, \widetilde{\frac{1+\alpha}{2}}, \frac{2+\alpha}{2}, \widetilde{\frac{2+\alpha}{2}}, \frac{1}{2}, \widetilde{\frac{1}{2}}, \widetilde{1}|q, q^{4+2\alpha}(1-q)^4 x^2 t^2) - \frac{q^{1+\alpha}(1-q)xt}{1-q^{1+\alpha}} \times \\ &{}_0\phi_7(-; \frac{2+\alpha}{2}, \widetilde{\frac{2+\alpha}{2}}, \frac{3+\alpha}{2}, \widetilde{\frac{3+\alpha}{2}}, \frac{3}{2}, \widetilde{\frac{3}{2}}, \widetilde{1}|q, q^{8+2\alpha}(1-q)^4 x^2 t^2) \Bigg],\end{aligned} \tag{54}$$

and replacing t in (54) by it, we obtain

$$\sum_{n=0}^{\infty}\frac{L_{2n,q}^{(\alpha)}(x)(-t^2)^n}{\{1+\alpha\}_{2n,q}}+\frac{it}{\{1+\alpha\}_q}\sum_{n=0}^{\infty}\frac{L_{2n+1,q}^{(\alpha)}(x)(-t^2)^n}{\{2+\alpha\}_{2n,q}}$$
$$=(Cos_q(t)+iSin_q(t))_0\phi_7(-;\frac{1+\alpha}{2},\frac{\widetilde{1+\alpha}}{2},\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{1}{2},\frac{\widetilde{1}}{2},\widetilde{1}|$$
$$|q,-q^{4+2\alpha}(1-q)^4x^2t^2)+\frac{q^{1+\alpha}(1-q)xt}{1-q^{1+\alpha}}(Sin_q(t)-iCos_q(t))\times$$
$${}_0\phi_7(-;\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{3+\alpha}{2},\frac{\widetilde{3+\alpha}}{2},\frac{3}{2},\frac{\widetilde{3}}{2},\widetilde{1}|q,-q^{8+2\alpha}(1-q)^4x^2t^2). \tag{55}$$

Next equate real and imaginary parts from both sides to arrive at the generating functions

$$\sum_{n=0}^{\infty}\frac{L_{2n,q}^{(\alpha)}(x)(-t^2)^n}{\{1+\alpha\}_{2n,q}}=Cos_q(t)_0\phi_7(-;\frac{1+\alpha}{2},\frac{\widetilde{1+\alpha}}{2},\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{1}{2},\frac{\widetilde{1}}{2},\widetilde{1}|$$
$$|q,-q^{4+2\alpha}(1-q)^4x^2t^2)+\frac{q^{1+\alpha}(1-q)xt}{1-q^{1+\alpha}}Sin_q(t)$$
$${}_0\phi_7(-;\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{3+\alpha}{2},\frac{\widetilde{3+\alpha}}{2},\frac{3}{2},\frac{\widetilde{3}}{2},\widetilde{1}|q,-q^{8+2\alpha}(1-q)^4x^2t^2) \tag{56}$$

and

$$\sum_{n=0}^{\infty}\frac{L_{2n+1,q}^{(\alpha)}(x)(-t^2)^n}{\{2+\alpha\}_{2n,q}}=\frac{\{1+\alpha\}_qSin_q(t)}{t}{}_0\phi_7(-;\frac{1+\alpha}{2},\frac{\widetilde{1+\alpha}}{2},\frac{2+\alpha}{2},$$
$$,\frac{\widetilde{2+\alpha}}{2},\frac{1}{2},\frac{\widetilde{1}}{2},\widetilde{1}|q,-q^{4+2\alpha}(1-q)^4x^2t^2)-xq^{1+\alpha}Cos_q(t)\times$$
$${}_0\phi_7(-;\frac{2+\alpha}{2},\frac{\widetilde{2+\alpha}}{2},\frac{3+\alpha}{2},\frac{\widetilde{3+\alpha}}{2},\frac{3}{2},\frac{\widetilde{3}}{2},\widetilde{1}|q,-q^{8+2\alpha}(1-q)^4x^2t^2). \tag{57}$$

The following generating function is a q-analogue of [18, p.43, (74')], [8, p.399], [19, p.120 11"].

$$\sum_{n=0}^{\infty}L_{n,q}^{(\alpha-n)}(x)t^nq^{\binom{n}{2}-n\alpha}=\frac{E_{\frac{1}{q}}(-xt)}{(-t;q)_{-\alpha}},\ |t|<1,\ |x|<1. \tag{58}$$

Proof.

$$\sum_{n=0}^{\infty} L_{n,q}^{(\alpha-n)}(x)t^n q^{\binom{n}{2}-n\alpha} =$$

$$= \sum_{n=0}^{\infty} t^n q^{\binom{n}{2}-n\alpha} \sum_{k=0}^{n} \frac{\langle 1+\alpha-n;q\rangle_n \langle -n;q\rangle_k}{\langle 1+\alpha-n;q\rangle_k \langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k\alpha}(1-q)^k x^k}{\langle 1;q\rangle_n} =$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^{n+k} q^{\frac{n^2+2nk+k^2-n-k}{2}-(n+k)\alpha} \times$$

$$\frac{\langle 1+\alpha-n-k;q\rangle_{n+k} \langle -n-k;q\rangle_k}{\langle 1+\alpha-n-k;q\rangle_k \langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k\alpha}(1-q)^k x^k}{\langle 1;q\rangle_{n+k}} =$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^{n+k} q^{\frac{n^2+k^2-n-k}{2}-n\alpha} \frac{\langle 1+\alpha-n;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_n} (-1)^k (1-q)^k x^k =$$

$$= \sum_{n=0}^{\infty} t^n (-1)^n \frac{\langle -\alpha;q\rangle_n}{\langle 1;q\rangle_n} \sum_{k=0}^{\infty} (1-q)^k x^k (-1)^k \frac{t^k q^{\binom{k}{2}}}{\langle 1;q\rangle_k} = \frac{E_{\frac{1}{q}}(-xt)}{(-t;q)_{-\alpha}}. \tag{59}$$

□

3. Product expansions

The theory of commutative ordinary differential operators was first explored in depth by Burchnall and Chaundy [3], [4], [5]. This technique was then used to find differential equations for hypergeometric functions in many papers, e.g. [6]. Unfortunately, it is very difficult to find q-analogues of these results. We will however prove four q-products expansions. We begin with a q-analogue of Carlitz' result [7, p. 220].

Theorem 3.1. *Let ϵ denote the operator which maps $f(x)$ to $f(qx)$. Then*

$$\begin{aligned} L_{n,q,c}^{(\alpha)}(x) = \prod_{k=3}^{n} (q^k x D_q \epsilon^{-1} - xq^{2k+\alpha-1} + \{\alpha+k\}_q) \\ (qxD_q - xq^{3+\alpha} + \{\alpha+2\}_q)(xD_q - xq^{1+\alpha} + \{\alpha+1\}_q)1, \end{aligned} \tag{60}$$

where the number of factors to the right is n.

Proof. The theorem is true for $n = 0$. Also we find that it's true for $n = 1, 2$. Assume that it is true for $n-1$, $n \geq 3$. Then we must prove that

$$\sum_{k=0}^{n}\frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{-\binom{k}{2}+k^2+kn+\alpha k}(1-q)^k x^k}{(1-q)^n}=$$
$$=(q^n xD_q\epsilon^{-1}-xq^{2n+\alpha-1}+\{\alpha+n\}_q) \tag{61}$$
$$\sum_{k=0}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2-k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}}.$$

A calculation shows that RHS=

$$\sum_{k=0}^{n-1}\frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k}\frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2-k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n}-$$
$$-\sum_{k=0}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2-k}{2}+kn+\alpha k}q^{2n+\alpha-1}(1-q)^k x^{k+1}}{(1-q)^{n-1}}+$$
$$+q^n\sum_{k=1}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_{k-1}}\frac{q^{\frac{k^2-k}{2}+kn+\alpha k}q^{-k}(1-q)^k x^k}{(1-q)^n}. \tag{62}$$

Finally, we must prove that

$$\frac{1-q^{n+\alpha}}{1-q^{k+\alpha}}\frac{1-q^{-n}}{1-q^k}=\frac{q^{-k}(1-q^{n+\alpha})}{1-q^{\alpha+k}}\frac{1-q^{k-n}}{1-q^k}+\frac{q^{n-2k}(1-q^{k-n})}{1-q^{\alpha+k}}-q^{-2k+n}, \tag{63}$$

which is easily checked. □

The following theorem, which is a q-analogue of [25, p.374 (2)] is proved in a similar way.

Theorem 3.2.

$$L_{n,q,c}^{(\alpha)}(x)=E_{\frac{1}{q}}(x)\prod_{k=1}^{n}(q^{k+\alpha}xD_q+\{\alpha+k\}_q)E_q(-x). \tag{64}$$

Proof. The theorem is true for $n = 0$. Assume that it is true for $n-1$. Then we must prove that

$$\sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+kn+\alpha k}(1-q)^k x^k}{(1-q)^n} =$$
$$= E_{\frac{1}{q}}(x)(q^{n+\alpha} x D_q + \{\alpha+n\}_q)$$
$$\sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k(n-1)+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}} E_q(-x). \tag{65}$$

A calculation shows that RHS=$E_{\frac{1}{q}}(x)\times$

$$\Bigg[\{\alpha+n\}_q \sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k(n-1)+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}}$$
$$+q^{n+\alpha} x$$
$$\Bigg[\sum_{k=1}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+k(n-1)+\alpha k}(1-q)^k(1-q^k) x^{k-1}}{(1-q)^n}$$
$$-\sum_{k=0}^{n-1} \frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{-\binom{k}{2}+k^2+kn+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}}\Bigg]\Bigg] E_q(-x). \tag{66}$$

We must prove that

$$\frac{1-q^{n+\alpha}}{\langle 1+\alpha;q\rangle_k} \frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2}{2}+\frac{k}{2}+kn+\alpha k}(1-q)^k}{(1-q)^n} =$$
$$\frac{1-q^{n+\alpha}}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2}{2}-\frac{k}{2}+kn+\alpha k}(1-q)^k}{(1-q)^n} +$$
$$+\frac{q^{n+\alpha}}{\langle 1+\alpha;q\rangle_k} \frac{\langle 1-n;q\rangle_k}{\langle 1;q\rangle_k} \frac{q^{\frac{k^2}{2}-\frac{k}{2}+kn+\alpha k}(1-q^k)(1-q)^k}{(1-q)^n} -$$
$$-\frac{q^{n+\alpha}}{\langle 1+\alpha;q\rangle_{k-1}} \frac{\langle 1-n;q\rangle_{k-1}}{\langle 1;q\rangle_{k-1}} \frac{q^{\frac{k^2}{2}-\frac{k}{2}+kn-n+\alpha k-\alpha}(1-q)^k}{(1-q)^n}, \tag{67}$$

which implies that

$$\frac{1-q^{n+\alpha}}{1-q^{k+\alpha}}\frac{1-q^{-n}}{1-q^{k}}=\frac{q^{-k}(1-q^{n+\alpha})}{1-q^{\alpha+k}}\frac{1-q^{k-n}}{1-q^{k}}+\frac{q^{n+\alpha-k}(1-q^{k-n})}{1-q^{\alpha+k}}-q^{-k}, \tag{68}$$

which is easily checked. □

The following theorem is a q-analogue of Chak [9], see also Chatterjea [12].

Theorem 3.3.

$$L_{n,q,c}^{(\alpha)}(x)=x^{-\alpha-n-1}E_{\frac{1}{q}}(x)(x^2D_q)^n x^{\alpha+1}E_q(-x). \tag{69}$$

Proof. The theorem is true for $n=0$. Assume that it is true for $n-1$. Then we must prove that

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2+k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n}=x^{-\alpha-n-1}E_{\frac{1}{q}}(x)x^2\times\\&\times D_q\sum_{k=0}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2+k}{2}+k(n-1)+\alpha k}(1-q)^k x^{k+\alpha+n}}{(1-q)^{n-1}}\times\\&\times E_q(-x).\end{aligned} \tag{70}$$

A calculation shows that RHS=

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k}\times\\&\times\frac{q^{\frac{k^2+k}{2}+k(n-1)+\alpha k}(1-q)^k x^k}{(1-q)^{n-1}}(\{k+\alpha+n\}_q(1+(1-q)x)-x)\\&=\frac{\langle 1+\alpha;q\rangle_n}{(1-q)^n}-\frac{\langle -n+1;q\rangle_{n-1}}{\langle 1;q\rangle_{n-1}}q^{\frac{n^2-n}{2}+n^2+\alpha n}x^n\\&+\sum_{k=1}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n+1;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2+k}{2}+kn-k+\alpha k}x^k}{(1-q)^{n-k}}(1-q^{k+\alpha+n})-\\&-\sum_{k=1}^{n-1}\frac{\langle 1+\alpha;q\rangle_{n-1}}{\langle 1+\alpha;q\rangle_{k-1}}\frac{\langle -n+1;q\rangle_{k-1}}{\langle 1;q\rangle_{k-1}}\frac{q^{\frac{k^2-k}{2}+kn+\alpha k}x^k}{(1-q)^{n-k}}=LHS.\end{aligned} \tag{71}$$

□

The following theorem is a q-analogue of Chatterjea [10] and a generalization of (69).

Theorem 3.4.

$$L_{n,q,c}^{(\alpha)}(x) = x^{-\alpha-n-k} E_{\frac{1}{q}}(x)(\{1-k\}_q x + q^{1-k} x^2 D_q)^n x^{\alpha+k} E_q(-x). \quad (72)$$

With the help of (69) we can prove a q-analogue of a bilinear generating formula for Laguerre polynomials of Chatterjea [12, p.57].

Theorem 3.5.

$$\sum_{n=0}^{\infty} \{n\}_q! L_{n,q}^{(\alpha-n)}(x) L_{n,q}^{(\beta-n)}(y) t^n =$$
$$E_{\frac{1}{q}}(x) E_{\frac{1}{q}}(y) \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} x^r y^s}{\{r\}_q! \{s\}_q!} {}_3\phi_0(\infty, -r-\alpha, -s-\beta; -|q, \frac{tq^{r+\alpha+s+\beta}}{1-q}). \quad (73)$$

Proof.

$$LHS = \sum_{n=0}^{\infty} \frac{x^{-\alpha-1}}{\{n\}_q!} E_{\frac{1}{q}}(x)(x^2 D_{q,x})^n x^{\alpha-n+1} E_q(-x) y^{-\beta-1} \times$$
$$E_{\frac{1}{q}}(y)(y^2 D_{q,y})^n y^{\beta-n+1} E_q(-y) t^n = E_{\frac{1}{q}}(x) E_{\frac{1}{q}}(y) x^{-\alpha-1} y^{-\beta-1} \times$$
$$\sum_{n=0}^{\infty} \frac{t^n}{\{n\}_q!} (x\theta_1)^n (y\theta_2)^n x^{\alpha-n+1} y^{\beta-n+1} E_q(-x) E_q(-y)$$
$$= E_{\frac{1}{q}}(x) E_{\frac{1}{q}}(y) x^{-\alpha-1} y^{-\beta-1} \sum_{n=0}^{\infty} \frac{t^n}{\{n\}_q!} (x\theta_1)^n (y\theta_2)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r\}_q!} x^{\alpha+r-n+1}$$
$$\sum_{s=0}^{\infty} \frac{(-1)^s}{\{s\}_q!} y^{\beta+s-n+1} = E_{\frac{1}{q}}(x) E_{\frac{1}{q}}(y) x^{-\alpha-1} y^{-\beta-1}$$
$$\sum_{n=0}^{\infty} \frac{t^n}{\{n\}_q!} \sum_{r=0}^{\infty} \frac{(-1)^r}{\{r\}_q!} \{r+\alpha-n+1\}_{n,q} x^{\alpha+r+1}$$
$$\sum_{s=0}^{\infty} \frac{(-1)^s}{\{s\}_q!} \{s+\beta-n+1\}_{n,q} y^{\beta+s+1} = E_{\frac{1}{q}}(x) E_{\frac{1}{q}}(y) \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} x^r y^s}{\{r\}_q! \{s\}_q!}$$
$$\sum_{n=0}^{\infty} \frac{\langle -r-\alpha, -s-\beta; q\rangle_n}{\langle 1; q\rangle_n (1-q)^n} q^{-2\binom{n}{2} + n(\alpha+r+\beta+s)} t^n = RHS. \quad (74)$$

□

By the same method, we can find a q-analogue of a bilinear generating formula for Laguerre polynomials of Chatterjea [11, p.88].

Theorem 3.6.

$$\sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n(xyt)^n}{\langle\alpha+1,\beta+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\beta)}(y)=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\times$$
$$\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}{}_3\phi_2(\gamma,\alpha+r+1,\beta+s+1;\alpha+1,\beta+1|q,xyt). \tag{75}$$

Proof.

$$LHS=\sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n}{\langle\alpha+1,\beta+1;q\rangle_n}\frac{x^{-\alpha-1}}{(\{n\}_q!)^2}E_{\frac{1}{q}}(x)(x^2D_{q,x})^n x^{\alpha+1}E_q(-x)\times$$
$$y^{-\beta-1}E_{\frac{1}{q}}(y)(y^2D_{q,y})^n y^{\beta+1}E_q(-y)t^n=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)x^{-\alpha-1}y^{-\beta-1}\times$$
$$\sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n t^n}{\langle\alpha+1,\beta+1;q\rangle_n(\{n\}_q!)^2}(x\theta_1)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{\{r\}_q!}x^{\alpha+r+1}(y\theta_2)^n\sum_{s=0}^{\infty}\frac{(-1)^s}{\{s\}_q!}\times$$
$$y^{\beta+s+1}=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{n=0}^{\infty}\frac{\langle 1,\gamma;q\rangle_n t^n}{\langle\alpha+1,\beta+1;q\rangle_n(\{n\}_q!)^2}\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^{n+r}y^{n+s}}{\{r\}_q!\{s\}_q!}\times$$
$$\{r+\alpha+1\}_{n,q}\{s+\beta+1\}_{n,q}=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}\times$$
$$\sum_{n=0}^{\infty}\frac{\langle\gamma,\alpha+r+1,\beta+s+1;q\rangle_n}{\langle 1,\alpha+1,\beta+1;q\rangle_n}(xyt)^n=RHS. \tag{76}$$

□

Put $\gamma=\beta+1$ in (75) to obtain

Theorem 3.7.

$$\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\beta)}(y)=E_{\frac{1}{q}}(y)\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s}$$
$$\times\,{}_1\phi_2(\beta+s+1;\alpha+1|q,-xt(1-q)q^{1+\alpha}||-;tq^{\beta+s+1}). \tag{77}$$

Proof.

$$\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\beta)}(y)=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}\times$$
$$_2\phi_1(\alpha+r+1,\beta+s+1;\alpha+1|q,t)=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}\times$$
$$\frac{1}{(t;q)_{\beta+s+1}}{}_2\phi_2(\beta+s+1,-r;\alpha+1|q,tq^{\alpha+r+1}||-;tq^{\beta+s+1})=$$
$$E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!}\times$$
$$_2\phi_2(\beta+s+1,-r;\alpha+1|q,tq^{\alpha+r+1}||-;tq^{\beta+s+1})=E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)$$
$$\times\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!}\sum_{k=0}^{r}\frac{\langle\beta+s+1,-r;q\rangle_k}{\langle 1,\alpha+1;q\rangle_k}$$
$$\times\frac{(-t)^k q^{\binom{k}{2}+k(\alpha+r+1)}}{(tq^{\beta+s+1};q)_k}=E_{\frac{1}{q}}(y)\frac{1}{(t;q)_{\beta+1}}\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\beta+1};q)_s}$$
$$\times\sum_{k=0}^{\infty}\frac{\langle\beta+s+1;q\rangle_k(-1)^k}{\langle 1,\alpha+1;q\rangle_k}\frac{(xt)^k(1-q)^k q^{k^2+k\alpha}}{(tq^{\beta+s+1};q)_k}=RHS. \tag{78}$$

□

Put $\beta=\alpha$ and $\gamma=\alpha+1$ in (75) to obtain the following q-analogue of the Hardy-Hille formula

Theorem 3.8.

$$\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\alpha)}(y)=\frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\times$$
$$\sum_{s,r,k=0}^{\infty}\frac{(-y)^s}{\{s\}_q!}\frac{(-x)^r}{\{r\}_q!}\frac{(1-q)^{2k}(xyt)^k q^{\alpha k+k^2}}{\langle 1,\alpha+1;q\rangle_k(tq^{\alpha+1};q)_{r+2k+s}}. \tag{79}$$

Proof.

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\langle 1;q\rangle_n t^n}{\langle\alpha+1;q\rangle_n}L_{n,q}^{(\alpha)}(x)L_{n,q}^{(\alpha)}(y)=\\
&E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}{}_2\phi_1(\alpha+r+1,\alpha+s+1;\alpha+1,|q,t)=\\
&E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}x^r y^s}{\{r\}_q!\{s\}_q!}\frac{}{(t;q)_{\alpha+r+s+1}}\times\\
&{}_2\phi_1(-r,-s;\alpha+1,|q,tq^{\alpha+r+s+1})=\frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!(tq^{\alpha+1};q)_r}\times\\
&\sum_{s=0}^{\infty}\frac{(-y)^s}{\{s\}_q!(tq^{\alpha+1+r};q)_s}{}_2\phi_1(-r,-s;\alpha+1,|q,tq^{\alpha+r+s+1})=\\
&\frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!(tq^{\alpha+1};q)_r}\sum_{s,k=0}^{\infty}\frac{(-y)^{s+k}}{\{s+k\}_q!(tq^{\alpha+1+r};q)_{s+k}}\times\\
&\frac{\langle -s-k,-r;q\rangle_k}{\langle 1,\alpha+1;q\rangle_k}t^k q^{(\alpha+r+s+1)k+k^2}=\frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\sum_{r=0}^{\infty}\frac{(-x)^r}{\{r\}_q!(tq^{\alpha+1};q)_r}\times\\
&\sum_{s,k=0}^{\infty}\frac{(-y)^s(yt)^k(1-q)^{s+k}\langle -r;q\rangle_k q^{(\alpha+r)k+\frac{k^2}{2}+\frac{k}{2}}}{\langle 1;q\rangle_s\langle 1,\alpha+1;q\rangle_k(tq^{\alpha+1+r};q)_{s+k}}=\frac{E_{\frac{1}{q}}(x)E_{\frac{1}{q}}(y)}{(t;q)_{\alpha+1}}\times\\
&\sum_{s,r,k=0}^{\infty}\frac{(-y)^s}{\{s\}_q!}\frac{(-x)^r}{\{r\}_q!}\frac{(1-q)^{2k}(xyt)^k q^{\alpha k+k^2}}{\langle 1,\alpha+1;q\rangle_k(tq^{\alpha+1};q)_{r+2k+s}}.
\end{aligned}
\tag{80}
$$

□

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