q-BERNOULLI AND q-EULER POLYNOMIALS, AN UMBRAL APPROACH II

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Abstract We proceed with pseudo-q-Appell polynomials in the spirit of [12]. It turns out that these q-Bernoulli numbers are the same as $\mathrm{B}_{\mathrm{JHC},\nu,q}$. As in [12] we find q-analogues of many formulas in [38], the umbral calculus works remarkably well also for pseudo-q-Appell pol., only the q is put up instead of down corresponding to inversion of basis. We also find new q-Euler-Maclaurin expansions.

1. Historical introduction

This is a continuation of [12]. The aim is to present pseudo-q-Appell polynomials and to give the corresponding formulas for pseudo-q-Bernoulli and pseudo-q-Euler polynomials. In this chapter some historical aspects of umbral calculus as well as finite differences are given. The purpose is to point out the close connection between these two subjects. The related Stirling numbers are also discussed. We also present two new q-Euler-Maclaurin formulas together with the most important q-Appell polynomial formulas from [12]. In the second chapter we come to the core of the article, the pseudo-q-Appell polynomials, which show a remarkable resemblance to the q-Appell polynomials.

The Bernoulli- and Stirling numbers are intimately connected by a well-known formula, they complement each other. The Stirling numbers were probably first used by Thomas Harriot (c. 1560-1621), a British astronomer and mathematician, who seldom published his writings due to financial insecurity. In 1715 Brook Taylor (1685-1731) used calculus of finite differences in his monumental work *Methodus Incrementorum Directa et Inversa*. Taylor travelled to France to see Pierre Rémond de Montmort (1678-1719), who had published on series with inverse factorial function argument, a precursor to the q-binomial theorem.

James Stirling (1692-1770) in his most important work *Methodus Differentialis* [45] gave a treatise on the calculus of finite differences.

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Here we can find the series

$$\frac{1}{x-a} = \frac{1}{x} + \frac{a}{x(x+1)} + \frac{a(a+1)}{x(x+1)(x+2)} + \dots, \tag{1}$$

usually called Stirling's series, but given before by Montmort and François Nicole (1683-1758) [35] in 1717 [49, p. 32]. This formula for a slowly converging series was later q-deformed by F.-H. Jackson.

By this time the Bernoulli numbers were well-known in England; around 1735, Colin Maclaurin (1698-1746) and Euler independently discovered the Euler-Maclaurin summation formula. There are more than one form of this formula. The form I is the one found in Saalschütz [44], with q-version in [12]. In this paper we find two q-analogues of Nørlund's form II. Finally, there is Malmsten's form III, which is similar to II.

Already in 1803 Robert Woodhouse (1773-1827) had attempted to put the calculus on a rigorous algebraic foundation using a formal series expansions method similar to that developed by Lagrange in his important book *Principles of Analytic Calculation*.

The first immediate reaction on Woodhouse 1803 book came when Rev. John Brinkley (1763-1835) in a paper in *Phil. Trans. Royal Soc. London* in 1807 started the first symbolic calculus. The Brinkley paper contained some abbreviations for expressions like $\frac{x^n}{n!}$ and $\frac{\dot{x}}{n!}$. Here $\dot{x}$ is the fluxion of x. Brinkley also calculated with expressions for differences of nothing, the precursor of Stirling numbers. Of course the Stirling numbers were known already to Thomas Harriot, but Brinkley probably knew nothing about this. Although Brinkley knew about Arbogast, he writes: My publication has hitherto been delayed by my unwillingness to offer a fluxional notation different from either that of Newton or Leibniz, each of which is very inconvenient as far as regards the application of the theorems for finding fluxions.

Brinkley's work became widely known in Russia, partly due to his fame as astronomer. After having acquired the chair of astronomy at Trinity College, Dublin, in 1790, Brinkley had to wait 18 years until the new telescope was erected, where it still stands. Brinkley was eighteen years waiting for his telescope, and he had eighteen years more in which to use it. During the first of these periods Brinkley devoted himself to mathematical research; during the latter he became a celebrated astronomer.

The Lucas umbral calculus was widespread in Russia, for example you will find the formula defining the Bernoulli numbers in Chistiakov 1895 [7]. The work of Blissard on umbral calculus also had a stir in

Russia. In [7, p. 113] you can find the Blissard Bernoulli number formulas with sine and cosine.

We are now going to turn to how Nørlund happened to get influences to his article [37] and to his book [38], published in French 1920 and in German 1924.

In 1898 Grigoriew [22, p.147] defined Bernoulli numbers of higher order as follows

$$\frac{t^n}{(\mathrm{e}^t - 1)^n}\mathrm{e}^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathrm{B}_\nu^{(n)}(x)}{\nu!}, \ |t| < 2\pi. \tag{2}$$

This corresponds to (31).

The Stirling numbers had also been published in [22, p.187] under different name. Grigoriew was influenced by Sintsov [46], Imchenetsky [25] and Mikhail Egorovich Vashchenko-Zakharchenko (1825 - 1912).

Weierstrass has said that the finite differences will once play a leading rôle in mathematics. Two important elements of finite differences are the Bernoulli numbers and the Γ function. Nørlund says in a letter to Mittag-Leffler 1919 "Someone who is not an expert in these areas, may not be expert on finite differences".

Nørlund had discussed the equation $\triangle f(x) = D\varphi(x)$ many years in his correspondance with Mittag-Leffler.

Theorem 1.1. *A q-analogue of Nørlund* [37, p. 125].
The following dual Euler-Maclaurin formulas apply:

$$D_q f(x \oplus_q y) \ddot{=} \sum_{k=0}^{\nu} \frac{\mathrm{B}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \triangle_{\mathrm{NWA},q} D_q^k f(y). \tag{3}$$

$$D_q f(x \oplus_q y) \ddot{=} \sum_{k=0}^{\nu} \frac{\mathrm{B}_{\mathrm{JHC},k,q}(x)}{\{k\}_q!} \triangle_{\mathrm{JHC},q} D_q^k f(y). \tag{4}$$

Proof. We only prove the first formula. Let $\varphi(x)$ be a polynomial of degree ν which satisfies the equation

$$\triangle_{\mathrm{NWA},q} f(x) = D_q \varphi(x). \tag{5}$$

We can rewrite this slightly.

$$\triangle_{\mathrm{NWA},q} f(x) = \varphi(x \oplus_q B_{\mathrm{NWA},q} \oplus_q 1). \tag{6}$$

Because of a well-known umbral formula we have

$$f(x \oplus_q y) = \varphi(y \oplus_q B_{\text{NWA},q}(x)) \ddot{=} \sum_{k=0}^{\infty} \frac{\text{B}_{\text{NWA},k,q}(x)}{\{k\}_q!} D_q^k \varphi(y) \ddot{=} \\ \sum_{k=0}^{\nu} \frac{\text{B}_{\text{NWA},k,q}(x)}{\{k\}_q!} \triangle_{\text{NWA},q} D_q^{k-1} f(y). \tag{7}$$

Finally operate on both sides with D_q. □

Carl Johan Malmsten (1814-1886), contemporary and friend of Gösta Mittag-Leffler (1846-1927), discussed a similar formula ($q = 1$) in [30] and [31].

2. Pseudo q-Appell polynomials

We will now start to introduce the notation associated with pseudo-q-Appell polynomials. Often a formula in [12] is slightly changed by putting the q up instead of down. This corresponds to inversion of the basis. In some formulas we keep q down, but change it to $\frac{1}{q}$.

Definition 1. The q-coaddition is defined by

$$(a \oplus^q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-n)} a^k b^{n-k}, \; n = 0, 1, 2, \ldots. \tag{8}$$

Definition 2. The Ward-Alsalam q-coshift operator is given by

$$E(\oplus^q)^{\omega}(x^n) \equiv (x \oplus^q \omega)^n \tag{9}$$

When $\omega = 1$, we denote this operator $E(\oplus^q)$.

Definition 3. The invertible linear difference operator for the NWA, is defined by

$$\underset{\omega}{\triangle^q_{\text{NWA}}} \equiv \frac{E(\oplus^q)^{\omega} - I}{\omega}, \; \omega \in \mathbb{C}, \tag{10}$$

where I is the identity operator. When $\omega = 1$, we denote this operator $\triangle^q_{\text{NWA}}$.

Definition 4. A q-analogue of the mean value operator of Jordan [27, p. 6] ($\omega = 1$), Nørlund [38, p. 3], and [33, p. 30].

$$\underset{\omega}{\nabla^q_{\text{NWA}}} \equiv \frac{E(\oplus^q)^{\omega} + I}{2}. \tag{11}$$

When $\omega = 1$, we denote this operator ∇^q_{NWA}.

Definition 5. If ω is a Ward number $\overline{n}_q$, the difference operator for the NWA is defined by

$$\underset{\overline{n}_q}{\triangle^q_{\text{NWA}}} \equiv \frac{E(\oplus^q)^{\overline{n}_q} - I}{n}. \tag{12}$$

Theorem 2.1. *A q-analogue of the Newton-Gregory series* [6, p. 21, 2.7], [27, p. 26], [33, 2.5.1], [29, p. 243].

$$f(\bar{n}_{\frac{1}{q}}) = \sum_{k=0}^{n} \binom{n}{k} (\triangle^q_{\text{NWA}})^k f(0). \tag{13}$$

The formula (13) can be inverted as follows

Theorem 2.2. *A q-analogue of* [29, p. 136, (3)], [33, 2.5.2].

$$(\triangle^q_{\text{NWA}})^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x \oplus^q \bar{k}_{\frac{1}{q}}). \tag{14}$$

The pseudo-q-Appell polynomials are characterized by the factor $\mathrm{E}_{\frac{1}{q}}(xt)$ on the left hand side in the definition of generating function.

In the spirit of Milne-Thomson [33, p. 125-147], which we will follow closely, we will call these q-polynomials Φ^q.

Examples of pseudo-q-Appell polynomials or Φ^q polynomials are $\mathrm{B}^{(n),q}_{\text{NWA},\nu}(x)$ and $F^{(n),q}_{\text{NWA},\nu}(x)$.

Definition 6. A q-analogue of [33, p. 124]. For every power series $f_n(t)$, the Φ_q polynomials of degree ν and order n have the following generating function

$$f_n(t)\mathrm{E}_{\frac{1}{q}}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n),q}_\nu(x). \tag{15}$$

By putting $x = 0$, we have

$$f_n(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n),q}_\nu, \tag{16}$$

where $\Phi^{(n),q}_\nu$ is called a Φ_q number of degree ν and order n.

It will be convenient to fix the value for $n = 0$ and $n = 1$.

$$\Phi^{(0),q}_\nu(x) \equiv x^\nu;\ \Phi^{(1),q}_\nu(x) \equiv \Phi^q_\nu(x). \tag{17}$$

Theorem 2.3. *A q-analogue of* [2], [33, p. 125 (4), (5)]

$$D_q \Phi^{(n),q}_\nu(x) = \{\nu\}_q \Phi^{(n),q}_{\nu-1,q}(qx). \tag{18}$$

$$\int_a^x \Phi^{(n),q}_\nu(qt)\, d_q(t) = \frac{\Phi^{(n),q}_{\nu+1,q}(x) - \Phi^{(n),q}_{\nu+1,q}(a)}{\{\nu+1\}_q}. \tag{19}$$

We obtain the two q-Taylor formulas

Theorem 2.4.

$$\Phi_{\nu}^{(n),q}(x \oplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{\binom{k}{2}} \Phi_{\nu-k,q}^{(n),q}(q^k x) y^k. \tag{20}$$

$$\Phi_{\nu}^{(n),q}(x \boxplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{2\binom{k}{2}} \Phi_{\nu-k,q}^{(n),q}(q^k x) y^k. \tag{21}$$

The first formula gives the symbolic equality

Theorem 2.5. *A q-analogue of* [33, p. 125 (3)]

$$\Phi_{\nu}^{(n),q}(x) \doteq (\Phi_q^{(n)} \boxplus_q x)^{\nu}. \tag{22}$$

Theorem 2.6. *A q-analogue of* [33, p. 125]

$$(\mathrm{E}_{\frac{1}{q}}(t) - 1) f_n(t) \mathrm{E}_{\frac{1}{q}}(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \triangle_{\mathrm{NWA}}^q \Phi_{\nu}^{(n),q}(x). \tag{23}$$

Proof. Operate on (15) with $\triangle_{\mathrm{NWA}}^q$. □

Theorem 2.7. *A q-analogue of* [33, p. 125]

$$\frac{(\mathrm{E}_{\frac{1}{q}}(t) + 1)}{2} f_n(t) \mathrm{E}_{\frac{1}{q}}(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \nabla_{\mathrm{NWA}}^q \Phi_{\nu}^{(n),q}(x). \tag{24}$$

Proof. Operate on (15) with ∇_{NWA}^q. □

A special case of the Φ_q polynomials are the β^q polynomials of degree ν and order n, which are obtained by putting $f_n(t) = \frac{t^n g(t)}{(\mathrm{E}_{\frac{1}{q}}(t)-1)^n}$ in (15).

Definition 7.

$$\frac{t^n g(t)}{(\mathrm{E}_{\frac{1}{q}}(t) - 1)^n} \mathrm{E}_{\frac{1}{q}}(xt) \equiv \sum_{\nu=0}^{\infty} \frac{t^{\nu} \beta_{\nu}^{(n),q}(x)}{\{\nu\}_q!}. \tag{25}$$

Theorem 2.8. *A q-analogue of* [33, (2), p. 126], [43, p. 704], [29, p. 240].

$$\triangle_{\mathrm{NWA}}^q \beta_{\nu}^{(n),q}(x) = \{\nu\}_q \beta_{\nu-1}^{(n-1),q}(x) = q D_q \beta_{\nu}^{(n-1),q}(q^{-1} x). \tag{26}$$

Proof. Use (23).

□

By (22) and (26) the following symbolic relations obtain.

Theorem 2.9. *A q-analogue of* [33, p. 126].

$$(\beta^{(n),q} \boxplus_q (x \oplus^q 1))^\nu - (\beta^{(n),q} \boxplus^q x)^\nu \doteq \{\nu\}_q (\beta^{(n-1),q} \boxplus_q x)^{\nu-1}. \tag{27}$$

$$(\beta^{(n),q} \boxplus_q 1)^\nu - \beta_\nu^{(n),q} \doteq \{\nu\}_q \beta_{\nu-1}^{(n-1),q}. \tag{28}$$

Theorem 2.10. *A q-analogue of* [37, (20), p. 163].

$$\triangle_{\text{NWA}}^q f(\beta_\nu^{(n),q}(x)) \equiv f(\beta_\nu^{(n),q}(x) \oplus_q 1) - f(\beta_\nu^{(n),q}(x)) = D_q f(\beta_\nu^{(n-1),q}(q^{-1}x)). \tag{29}$$

Theorem 2.11.

$$\sum_{k=1}^{\nu} \binom{\nu}{k}_q \beta_{\nu-k}^{(n),q}(q^k x) q^{\binom{k}{2}} = \{\nu\}_q \beta_{\nu-1}^{(n-1),q}(x). \tag{30}$$

Proof. Use (20) and (27). □

A special case of β^q polynomials are the generalized pseudo q-Bernoulli polynomials $\mathrm{B}_{\text{NWA},\nu}^{(n),q}(x)$ of degree ν and order n, which were defined for $q = 1$ in [33, p. 127], [37].

Definition 8. [38, (36) p. 132]. The generating function for $\mathrm{B}_{\text{NWA},\nu}^{(n),q}(x)$ is a q-analogue of [43, p. 704].

$$\frac{t^n}{(\mathrm{E}_{\frac{1}{q}}(t) - 1)^n} \mathrm{E}_{\frac{1}{q}}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathrm{B}_{\text{NWA},\nu}^{(n),q}(x)}{\{\nu\}_q!}, \ |t| < 2\pi. \tag{31}$$

This can be generalized to

Definition 9. Let $\{\omega_i\}_{i=1}^n \in \mathbb{C}$. The generating function for $\mathrm{B}_{\text{NWA},\nu}^{(n),q}(x|\omega_1, \ldots, \omega_n)$ is the following q-analogue of [38, (77) p. 143]:

$$\frac{t^n \omega_1 \ldots \omega_n}{\prod_{k=1}^n (\mathrm{E}_{\frac{1}{q}}(\omega_k t) - 1)} \mathrm{E}_{\frac{1}{q}}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{NWA,\nu}^{(n),q}(x|\omega_1, \ldots, \omega_n)}{\{\nu\}_q!}, \\ |t| < \min(|\frac{2\pi}{\omega_1}|, \ldots, |\frac{2\pi}{\omega_n}|). \tag{32}$$

Corollary 2.12. *A q-analogue of* [28, p. 639].

$$\mathrm{E}_q(t\mathrm{B}_{\text{NWA}}^q) \doteq \frac{t}{\mathrm{E}_{\frac{1}{q}}(t) - 1}. \tag{33}$$

The following special case is often used.

Definition 10. The pseudo NWA q-Bernoulli numbers are given by

$$B_{\text{NWA},n}^q \equiv \mathrm{B}_{\text{NWA},n}^{(1),q}. \tag{34}$$

By the generating function, it turns out that these q-Bernoulli numbers are the same as those previously studied:

$$\mathrm{B}^{q}_{\mathrm{NWA},\nu} = \mathrm{B}_{\mathrm{JHC},\nu,q} \tag{35}$$

The following recurrence obtains:

$$B^{q}_{\mathrm{NWA},0} = 1,\ (B^{q}_{\mathrm{NWA}} \boxplus_q\ 1)^k - B^{q}_{\mathrm{NWA},k} \doteq \delta_{1,k}. \tag{36}$$

We see immediately that the $B^{q}_{\mathrm{NWA}} \in \mathbb{Q}(q)$.

Theorem 2.13. *We have the following operational representation, a q-analogue of* [22, (4), p. 147], [47]*:*

$$\mathrm{B}^{(n),q}_{\mathrm{NWA},\nu}(\omega_1, \ldots, \omega_n) \doteq (\oplus^{n}_{\frac{1}{q},l=1}\omega_l \mathrm{B}^{q}_{\mathrm{NWA}})^{\nu}. \tag{37}$$

The following operator will be useful in connection with $\mathrm{B}^{(n),q}_{\mathrm{NWA},\nu}(x)$.

Definition 11. Compare [6, p. 32] ($n = 1$). The invertible operator $(S^{q}_{\mathrm{B,N}})^n \in \mathbb{C}(D_{\frac{1}{q}})$ is given by

$$(S^{q}_{\mathrm{B,N}})^n \equiv \frac{(\mathrm{E}_{\frac{1}{q}}(D_{\frac{1}{q}}) - I)^n}{D^{n}_{\frac{1}{q}}}. \tag{38}$$

This implies

Theorem 2.14.

$$(\triangle^{q}_{\mathrm{NWA}})^n = D^{n}_{\frac{1}{q}}(S^{q}_{\mathrm{B,N}})^n. \tag{39}$$

Theorem 2.15. *A q-analogue of* [39, p. 1225, i]. *The q-Bernoulli polynomials of degree ν and order n can be expressed as*

$$\mathrm{B}^{(n),q}_{\mathrm{NWA},\nu}(t) = (S^{q}_{\mathrm{B,N}})^{-n} t^{\nu}. \tag{40}$$

Proof.

$$LHS = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \mathrm{B}^{(n),q}_{\mathrm{NWA},k,q} t^{\nu-k} = \sum_{k=0}^{\infty} \frac{\mathrm{B}^{(n),q}_{\mathrm{NWA},k,q}}{\{k\}_q!} D^{k}_{q} t^{\nu} \overset{\text{by}(31)}{=} RHS. \tag{41}$$

□

Theorem 2.16. *A q-analogue of a generalization of* [6, p. 43, 3.3]

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathrm{B}^{(n),q}_{\mathrm{NWA},\nu}(x \oplus^{q} \overline{k}_{\frac{1}{q}}) = \{\nu - n + 1\}_{n,\frac{1}{q}} x^{\nu-n}. \tag{42}$$

Proof.

$$\begin{aligned}&LHS \overset{\text{by}(14)}{=} (\triangle_{\text{NWA}}^{q})^{n}\ \mathrm{B}_{\text{NWA},\nu}^{(n),q}(x) = D_{\frac{1}{q}}^{n}(S_{\text{B,N}}^{q})^{n}\ \mathrm{B}_{\text{NWA},\nu}^{(n),q}(x) = \\ &D_{\frac{1}{q}}^{n}(S_{\text{B,N}}^{q})^{n}(S_{\text{B,N}}^{q})^{-n}x^{\nu} = D_{\frac{1}{q}}^{n}x^{\nu} = \{\nu-n+1\}_{n,\frac{1}{q}}x^{\nu-n}.\end{aligned} \tag{43}$$

□

Corollary 2.17.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\mathrm{B}_{\text{NWA},\nu}^{(n),q}(\overline{k}_{\frac{1}{q}}) = \{n\}_{\frac{1}{q}}!\ \delta_{0,\nu-n}. \tag{44}$$

Proof. Put $x=0$ in (42). □

Theorem 2.18. *A q-analogue of* [29, (1), p. 240], [43, p. 699].

$$f(\mathrm{B}_{\text{NWA}}^{q} \boxplus_{q} (x \oplus^{q} 1)) - f(\mathrm{B}_{\text{NWA}}^{q} \boxplus_{q} x) \doteqdot D_{q}f(x), \tag{45}$$

where here and in the sequel, we have abbreviated the umbral symbol by $\mathrm{B}_{\text{NWA},q}$.

We will also state the corresponding equation for $\mathrm{B}_{\text{NWA},\nu}^{(n),q}$.

Theorem 2.19. *A q-analogue of* [22, (7), p. 152], [37, (11) p. 124], [38, (36) p. 132].

$$f(\mathrm{B}_{\text{NWA}}^{(n),q} \boxplus_{q} (x \oplus^{q} 1)) - f(\mathrm{B}_{\text{NWA}}^{(n),q} \boxplus_{q} x) \doteqdot D_{q}f(\mathrm{B}_{\text{NWA}}^{(n-1),q} \boxplus_{q} x). \tag{46}$$

Theorem 2.20. *Compare* [6, 3.15 p. 51]*, where the corresponding formula for Euler polynomials was given.*

$$B_{\text{NWA},\nu}^{q}(x) \equiv \frac{\{\nu\}_{q}}{\mathrm{E}_{\frac{1}{q}}(D_{\frac{1}{q}})-I}x^{\nu-1} = \frac{\{\nu\}_{q}}{E(\oplus^{q})-I}x^{\nu-1} \doteqdot (B_{\text{NWA}}^{q} \boxplus_{q} x)^{\nu}. \tag{47}$$

We will now follow Cigler [6] and give a few equations for pseudo q-Bernoulli polynomials. The first two of these equations are well-known in the literature ($q=1$).

Definition 12. A q-analogue of [24, p. 87], [6, p. 13], [50, p. 575].

$$s_{\text{NWA},m}^{q}(n) \equiv \sum_{k=0}^{n-1}(\overline{k}_{\frac{1}{q}})^{m},\ s_{\text{NWA},0}^{q}(1) \equiv 1. \tag{48}$$

Theorem 2.21. *A q-analogue of* [6, p. 13, p. 17: 1.11, p. 36], [29, p. 237].

$$s^{q}_{\mathrm{NWA},m}(n) = \frac{\mathrm{B}^{q}_{\mathrm{NWA},m+1}(\overline{(\overline{n_{\frac{1}{q}}})}) - \mathrm{B}^{q}_{\mathrm{NWA},m+1}}{\{m+1\}_q}$$
$$\equiv \frac{1}{\{m+1\}_q}\sum_{k=1}^{m+1}\binom{m+1}{k}_q (\overline{n_{\frac{1}{q}}})^k \mathrm{B}^{q}_{\mathrm{NWA},m+1-k}\; q^{k(k-m-1)} \tag{49}$$
$$\equiv \frac{1}{\{m+1\}_q}\sum_{k=0}^{m}\binom{m+1}{k}_q (\overline{n_{\frac{1}{q}}})^{m+1-k} \mathrm{B}^{q}_{\mathrm{NWA},k}\; q^{k(k-m-1)}.$$

Theorem 2.22. *A q-analogue of* [6, p. 45], [37, p. 127, (17)].

$$x^n = \int_x^{x\oplus_q 1} \mathrm{B}^{q}_{\mathrm{NWA},n}(qt)\, d_q(t) = \frac{\mathrm{B}^{q}_{\mathrm{NWA},n+1}(x\oplus_q 1) - \mathrm{B}^{q}_{\mathrm{NWA},n+1}(x)}{\{n+1\}_q}. \tag{50}$$

Proof. q-Integrate (18) for $n = 1$ and use (26). □

This can be rewritten as a q-analogue of the well-known identity [21, p. 496, 8.2].

$$x^n = \frac{1}{\{n+1\}_q}\sum_{k=0}^{n}\binom{n+1}{k}_{\frac{1}{q}} \mathrm{B}^{q}_{\mathrm{NWA},k}(x). \tag{51}$$

Cigler has given some examples of translation invariant operators. One of them is the Bernoulli operator.

Definition 13. The pseudo-q-Bernoulli operator is given by the following q-integral, a q-analogue of [6, p. 91], [8, p. 154], [42, p. 59], [43, p. 701, 703].

$$J^{q}_{\mathrm{B,N}} f(x) \equiv \int_x^{x\oplus_q 1} f(t)\, d_q(t). \tag{52}$$

Theorem 2.23. *A q-analogue of* [6, p. 44-45], [39, p. 1217].

The pseudo-q-Bernoulli operator can be expressed in the following form.

$$J^{q}_{\mathrm{B,N}} f(x) = \frac{\triangle^{q}_{\mathrm{NWA}}}{D_{\frac{1}{q}}} f(x). \tag{53}$$

Proof. Use (40) and (50). □

Theorem 2.24. *A q-analogue of* [6, p. 44-45]. *We can expand a given formal power series in terms of the* $\mathrm{B}^{q}_{\mathrm{NWA},k}(x)$ *as follows.*

$$f(x) = \sum_{k=0}^{\infty}\int_0^1 D^k_q f(t)\, d_q(t) \frac{\mathrm{B}^{q}_{\mathrm{NWA},k}(x)}{\{k\}_q!}. \tag{54}$$

Proof. Assume that

$$f(x)=\sum_{k=0}^{\infty}\frac{a_k}{\{k\}_q!}\mathrm{B}_{\mathrm{NWA},k}^{q}(x). \tag{55}$$

As we have

$$x^k=S_{\mathrm{B,N}}^{q}\mathrm{B}_{\mathrm{NWA},k}^{q}(x), \tag{56}$$

$$f(x)=\sum_{k=0}^{\infty}\frac{a_k}{\{k\}_q!}(S_{\mathrm{B,N}}^{q})^{-1}x^k \tag{57}$$

$$S_{\mathrm{B,N}}^{q}f(x)=\sum_{k=0}^{\infty}\frac{a_k}{\{k\}_q!}x^k. \tag{58}$$

This implies

$$a_k=D_q^kS_{\mathrm{B,N}}^{q}f(x)|_{x=0}=D_q^k\frac{\triangle_{\mathrm{NWA}}^{q}}{D_{\frac{1}{q}}}f(x)|_{x=0}=\int_0^1 D_q^kf(t)\,d_q(t). \tag{59}$$

□

A special case of the Φ^q polynomials are the pseudo η_q polynomials of order n, which are obtained by putting $f_n(t)=\frac{g(t)2^n}{(\mathrm{E}_{\frac{1}{q}}(t)+1)^n}$ in (15).

Definition 14. A q-analogue of [33, p. 142, (1)].

$$\frac{2^n}{(\mathrm{E}_{\frac{1}{q}}(t)+1)^n}g(t)\mathrm{E}_{\frac{1}{q}}(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}\eta_{\nu}^{(n)}{}^{q}(x)}{\{\nu\}_q!}. \tag{60}$$

By (24) we get a q-analogue of [33], [32, p. 519].

$$\nabla_{\mathrm{NWA}}^{q}\eta_{\nu}^{(n),q}(x)=\eta_{\nu}^{(n-1),q}(x). \tag{61}$$

We will now define the pseudo-q-Euler polynomials, a special case of the pseudo-η_q-polynomials.

Definition 15. The generating function for the pseudo-q-Euler polynomials of degree ν and order n $F_{\mathrm{NWA},\nu}^{(n),q}(x)$ is the following q-analogue of [41, p. 102], [33, p. 309], [48, p. 345].

$$\frac{2^n\mathrm{E}_{\frac{1}{q}}(xt)}{(\mathrm{E}_{\frac{1}{q}}(t)+1)^n}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x),,\ |t|<\pi. \tag{62}$$

This can be generalized to

Definition 16. Let $\{\omega_i\}_{i=1}^n \in \mathbb{C}$. The generating function for the pseudo-q-Euler polynomials of degree ν and order n $\mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x|\omega_1,\ldots,\omega_n)$ is the following q-analogue of [38, p. 143 (78)]:

$$\frac{2^n \mathrm{E}_{\frac{1}{q}}(xt)}{\prod_{k=1}^n (\mathrm{E}_{\frac{1}{q}}(\omega_k t)+1)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x|\omega_1,\ldots,\omega_n),$$
$$|t| < \min(|\frac{\pi}{\omega_1}|,\ldots,|\frac{\pi}{\omega_n}|). \tag{63}$$

Corollary 2.25.

$$\mathrm{E}_q(t\mathrm{F}_{\mathrm{NWA}}^q) \doteq \frac{2}{\mathrm{E}_{\frac{1}{q}}(t)+1}. \tag{64}$$

Obviously, $\mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$\mathrm{F}_{\mathrm{NWA},\nu}^{(1),q}(x|\omega) = \omega^\nu \mathrm{F}_{\mathrm{NWA},\nu}^q(\frac{x}{\omega}). \tag{65}$$

From

$$\underset{\omega_1,\ldots,\omega_n}{(\nabla_{\mathrm{NWA}}^q)^n} \mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x|\omega_1,\ldots,\omega_n) = x^\nu \tag{66}$$

we obtain

$$\underset{\omega_1,\ldots,\omega_p}{(\nabla_{\mathrm{NWA}}^q)^p} \mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x|\omega_1,\ldots,\omega_n) = \mathrm{F}_{\mathrm{NWA},\nu}^{(n-p),q}(x|\omega_{p+1},\ldots,\omega_n). \tag{67}$$

Theorem 2.26. *A q-analogue of* [33, p. 144, (7)], [38, (7), p.121].

$$\sum_{k=0}^{\nu} \binom{\nu}{k}_q \mathrm{F}_{\mathrm{NWA},\nu-k}^{(n),q}(x) + \mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x) = 2\mathrm{F}_{\mathrm{NWA},\nu}^{(n-1),q}(x). \tag{68}$$

With this formula we can compute all pseudo-q-Euler polynomials of order n, given knowledge of the polynomials of order $n-1$.

Definition 17. A q-analogue of [37, p. 139], [29, p. 252]. The first generalized q-Euler numbers are given by

$$\mathrm{F}_{\mathrm{NWA},\nu}^{(n),q} \equiv \mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(0). \tag{69}$$

Furthermore we put

$$\mathrm{F}_{\mathrm{NWA},k}^q \equiv \mathrm{F}_{\mathrm{NWA},k}^{(1),q};\ \mathrm{F}_{\mathrm{NWA},\nu}^q(x) \equiv \mathrm{F}_{\mathrm{NWA},\nu}^{(1),q}(x). \tag{70}$$

Theorem 2.27. *The operator expression is a q-analogue of* [6, 3.15 p. 51].

$$\mathrm{F}_{\mathrm{NWA},\nu}(x) \equiv \frac{2}{\mathrm{E}_{\frac{1}{q}}(D_{\frac{1}{q}})+I} x^\nu = \frac{2}{E(\oplus^q)+I} x^\nu \doteq (x \boxplus_q \mathrm{F}_{\mathrm{NWA}}^q)^\nu. \tag{71}$$

The following 2 recursion formulas are quite useful for the computations of the pseudo-q-Euler pol.

Theorem 2.28. *A q-analogue of* [38, (27), p. 24].

$$\mathrm{F}^{q}_{\mathrm{NWA},\nu}(x)+\sum_{k=0}^{\nu}\binom{\nu}{k}_{q}\mathrm{F}^{q}_{\mathrm{NWA},k}(x)=2x^{\nu}. \tag{72}$$

Theorem 2.29. *A q-analogue of* [6, 3.16 p. 51], [29, p. 252].

$$(\mathrm{F}^{q}_{\mathrm{NWA}}\boxplus_{q}1)^{n}+(\mathrm{F}^{q}_{\mathrm{NWA}})^{n}\doteq 2\delta_{0,n}. \tag{73}$$

Theorem 2.30. *A q-analogue of* [5, p. 6 (4.3)], [37, (19), p. 136], *a corrected version of* [29, p. 261].

$$f(\mathrm{F}^{q}_{\mathrm{NWA}}\boxplus_{q}(x\oplus^{q}1))+f(\mathrm{F}^{q}_{\mathrm{NWA}}\boxplus_{q}x)\doteq 2f(x). \tag{74}$$

We will also state the corresponding equation for $\mathrm{F}^{(n),q}_{\mathrm{NWA},\nu}$ written in two different forms.

Theorem 2.31. *A q-analogue of* [37, (19), p. 150, p. 155], [38, (29) p. 126].

$$\begin{gathered}\nabla^{q}_{\mathrm{NWA}}f(\mathrm{F}^{(n),q}_{\mathrm{NWA}}\boxplus_{q}x)\doteq f(\mathrm{F}^{(n-1),q}_{\mathrm{NWA}}\boxplus_{q}x)\doteq\\ \nabla^{q}_{\mathrm{NWA}}f(\mathrm{F}^{(n),q}_{\mathrm{NWA}}(x))\doteq f(\mathrm{F}^{(n-1),q}_{\mathrm{NWA}}(x)).\end{gathered} \tag{75}$$

As before we have

$$\mathrm{F}^{q}_{\mathrm{NWA},n}=\mathrm{F}_{\mathrm{JHC},n,q}. \tag{76}$$

Theorem 2.32. *We have the following operational representation, a q-analogue of* [47].

$$\mathrm{F}^{(n),q}_{\mathrm{NWA},\nu}(\omega_{1},\ldots,\omega_{n})\doteq(\oplus^{n}_{\frac{1}{q},l=1}\omega_{l}\mathrm{F}^{q}_{\mathrm{NWA},l})^{\nu}. \tag{77}$$

Theorem 2.33. *The pseudo-q-Euler pol. can be expressed as a finite sum of diff. operators on x^n. Almost a q-analogue of* [27, p. 289].

$$\mathrm{F}^{q}_{\mathrm{NWA},n}(x)=\sum_{m=0}^{n}\frac{(-1)^{m}}{2^{m}}{\triangle^{q}_{\mathrm{NWA}}}^{m}x^{n}. \tag{78}$$

Theorem 2.34. *A generalization of (72).*

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}\mathrm{F}^{(n),q}_{\mathrm{NWA},\nu}(x\oplus^{q}\overline{k}_{\frac{1}{q}})=x^{\nu}. \tag{79}$$

Proof. Develop $(\nabla^{q}_{\mathrm{NWA}})^{n}\mathrm{F}^{(n),q}_{\mathrm{NWA},\nu}(x)$. □

Definition 18. A q-analogue of [24, p. 88]. The notation from N. Nielsen (1865–1925) [36, p. 401] is a slightly modified variant of the original paper by Lucas [29].

$$\sigma^{q}_{\mathrm{NWA},m}(n) \equiv \sum_{k=0}^{n-1}(-1)^{k}(\overline{k_{\frac{1}{q}}})^{m}. \tag{80}$$

Theorem 2.35. *A q-analogue of* [6, p. 53], [33, p. 307], [20, p. 136].

$$\sigma^{q}_{\mathrm{NWA},m}(n) = \frac{(-1)^{n-1}\mathrm{F}^{q}_{\mathrm{NWA},m}(\overline{n}_{\frac{1}{q}}) + \mathrm{F}^{q}_{\mathrm{NWA},m}}{2}. \tag{81}$$

Proof.

$$\begin{aligned} LHS &= \sum_{k=0}^{n-1}(-1)^{k}\nabla^{q}_{\mathrm{NWA}}\mathrm{F}^{q}_{\mathrm{NWA},m}(\overline{k_{\frac{1}{q}}}) = \\ &\frac{1}{2}\sum_{k=0}^{n-1}(-1)^{k}(\mathrm{F}^{q}_{\mathrm{NWA},m}(\overline{k_{\frac{1}{q}}} \oplus^{q} 1) + \mathrm{F}^{q}_{\mathrm{NWA},m}(\overline{k_{\frac{1}{q}}})) = RHS. \end{aligned} \tag{82}$$

□

So far we considered only pseudo-q-Bernoulli polynomials and pseudo-q-Euler polynomials of positive order n. As the sequel shows, it will be useful to allow n also to be a negative integer. The following calculations are q-analogues of Nørlund [38, p. 133 ff]

Definition 19. As a q-analogue of [38, (50) p. 133], we define pseudo-q-Bernoulli polynomials of two variables as

$$\begin{aligned} &\mathrm{B}^{(n+p),q}_{\mathrm{NWA},\nu}(x \oplus^{q} y|\omega_1,\ldots,\omega_{n+p}) \equiv \\ &(\mathrm{B}^{(n),q}_{\mathrm{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus^{q} \mathrm{B}^{(p),q}_{\mathrm{NWA}}(y|\omega_{n+1},\ldots,\omega_{n+p}))^{\nu}, \end{aligned} \tag{83}$$

where we assume that n and p operate on x and y respectively, and the same for any pseudo-$q-$polynomial.

The relation (83) shows that $\mathrm{B}^{(n),q}_{\mathrm{NWA},\nu}(x|\omega_1,\ldots,\omega_n)$ is a homogeneous function of $x,\omega_1,\ldots,\omega_n$ of degree ν, a q-analogue of [38, p. 134 (55)], i.e.

$$\mathrm{B}^{(n),q}_{\mathrm{NWA},\nu}(\lambda x|\lambda\omega_1,\ldots,\lambda\omega_n) = \lambda^{\nu}\mathrm{B}^{(n),q}_{\mathrm{NWA},\nu}(x|\omega_1,\ldots,\omega_n),\ \lambda \in \mathbb{C}. \tag{84}$$

And the same for pseudo-$q-$Euler polynomials.

This can be generalized as follows.

Theorem 2.36. *A q-analogue of* [43, p. 704], [38, p. 133]. *If* $\sum_{l=1}^{s} n_l = n$,

$$\mathrm{B}_{\mathrm{NWA},k}^{(n),q}(x_1 \oplus^q \ldots \oplus^q x_s) = \sum_{m_1+\ldots+m_s=k} \binom{k}{m_1,\ldots,m_s}_{\frac{1}{q}} \prod_{j=1}^{s} \mathrm{B}_{\mathrm{NWA},m_j}^{(n_j),q}(x_j). \tag{85}$$

where we assume that n_j operates on x_j. And the same for any pseudo-$q-$polynomial.

Proof. In umbral notation we have, as in the classical case

$$(x_1 \oplus^q \ldots \oplus^q x_s \oplus^q \overline{n}_{\frac{1}{q}}\gamma)^k \sim ((x_1 \oplus^q \overline{n_1}_{\frac{1}{q}}\gamma') \oplus^q \ldots \oplus^q (x_s \oplus^q \overline{n_s}_{\frac{1}{q}}\gamma''))^k, \tag{86}$$

where $\gamma', \ldots, \gamma''$ are distinct umbrae, each equivalent to γ. □

By (26) and (61) we get

$$(\triangle_{\mathrm{NWA}}^q)^n \mathrm{B}_{\mathrm{NWA},\nu}^{(n),q}(x) = \frac{\{\nu\}_q!}{\{\nu-n\}_q!} x^{\nu-n},$$

$$(\nabla_{\mathrm{NWA}}^q)^n \mathrm{F}_{\mathrm{NWA},\nu}^{(n),q}(x) = x^{\nu},$$

and we have

Definition 20. A q-analogue of [37, p. 177], [38, (66), p. 138]. The pseudo-q-Bernoulli polynomials of negative order $-n$ are given by

$$\mathrm{B}_{\mathrm{NWA},\nu}^{(-n),q}(x|\omega_1,\ldots,\omega_n) \equiv \frac{\{\nu\}_q!}{\{\nu+n\}_q!} \mathop{(\triangle_{\mathrm{NWA}}^q)^n}_{\omega_1,\ldots,\omega_n} x^{\nu+n}, \tag{87}$$

and the q-Euler polynomial of negative order $-n$ by the following q-analogue of [38, (67) p. 138]

$$\mathrm{F}_{\mathrm{NWA},\nu}^{(-n),q}(x|\omega_1,\ldots,\omega_n) \equiv \mathop{(\nabla_{\mathrm{NWA}}^q)^n}_{\omega_1,\ldots,\omega_n} x^{\nu}, \tag{88}$$

where $\nu, n \in \mathbb{N}$. This defines pseudo-q-Bernoulli- and pseudo-q-Euler polynomials of negative order as iterated $\triangle_{\mathrm{NWA}}^q$ and ∇_{NWA}^q operating on positive integer powers of x.

Furthermore,

$$\mathrm{B}_{\mathrm{NWA},\nu}^{(-n),q} \equiv \mathrm{B}_{\mathrm{NWA},\nu}^{(-n),q}(0), \tag{89}$$

$$\mathrm{F}_{\mathrm{NWA},\nu}^{(-n),q} \equiv \mathrm{F}_{\mathrm{NWA},\nu}^{(-n),q}(0). \tag{90}$$

A calculation shows that formulas (26) and (61) hold for negative orders too, and we get

$$\mathrm{B}_{\mathrm{NWA},\nu}^{(-n-p),q}(x \oplus^q y) \doteq (\mathrm{B}_{\mathrm{NWA}}^{(-n),q}(x) \oplus^q \mathrm{B}_{\mathrm{NWA}}^{(-p),q}(y))^{\nu}, \tag{91}$$

and the same for pseudo-q-Euler polynomials.

A special case is the following q-analogue of [38, p. 139, (71)]:

$$\mathrm{B}_{\mathrm{NWA},\nu}^{(-n),q}(x \oplus^q y) \doteq (\mathrm{B}_{\mathrm{NWA}}^{(-n),q}(x) \oplus^q y)^\nu, \tag{92}$$

and the same for pseudo-q-Euler polynomials.

Theorem 2.37. *A recurrence formula for the pseudo-q-Bernoulli numbers and a recurrence formula for the pseudo-q-Euler numbers.*

If $n, p \in \mathbb{Z}$ *then*

$$\mathrm{B}_{\mathrm{NWA},\nu}^{(n+p),q} \doteq (\mathrm{B}_{\mathrm{NWA},q}^{(n),q} \oplus^q \mathrm{B}_{\mathrm{NWA}}^{(p),q})^\nu, \tag{93}$$

$$\mathrm{F}_{\mathrm{NWA},\nu}^{(n+p),q} \doteq (\mathrm{F}_{\mathrm{NWA}}^{(n),q} \oplus^q \mathrm{F}_{\mathrm{NWA}}^{(p),q})^\nu. \tag{94}$$

Theorem 2.38. *A q-analogue of* [38, p. 140 (72), (73)], [39, p. 1226, xvii].

$$(x \oplus^q y)^\nu \doteq (\mathrm{B}_{\mathrm{NWA}}^{(-n),q}(x) \oplus^q \mathrm{B}_{\mathrm{NWA}}^{(n),q}(y))^\nu, \tag{95}$$

$$(x \oplus^q y)^\nu \doteq (\mathrm{F}_{\mathrm{NWA}}^{(-n),q}(x) \oplus^q \mathrm{F}_{\mathrm{NWA}}^{(n),q}(y))^\nu. \tag{96}$$

Proof. Put $p = -n$ in (91). □

In particular for $y = 0$, we get a q-analogue of [39, p. 1226, xviii].

$$x^\nu \doteq (\mathrm{B}_{\mathrm{NWA}}^{(-n),q} \oplus^q \mathrm{B}_{\mathrm{NWA}}^{(n),q}(x))^\nu, \tag{97}$$

$$x^\nu \doteq (\mathrm{F}_{\mathrm{NWA}}^{(-n),q} \oplus^q \mathrm{F}_{\mathrm{NWA}}^{(n),q}(x))^\nu. \tag{98}$$

These recurrence formulas express pseudo-q-Bernoulli- and q-Euler polynomials of order n without mentioning polynomials of negative order.

As before, the pseudo-q-Bernoulli- and pseudo-q-Euler polynomials satisfy linear q-difference equations with constant coefficients.

The following theorem is useful for the computation of q-Bernoulli- and q-Euler polynomials of positive order. This is because the polynomials of negative order are of simpler nature and can easily be computed. When the $\mathrm{B}_{\mathrm{NWA},s,q}^{(-n)}$ etc. are known, (99) can be used to compute the $\mathrm{B}_{\mathrm{NWA},s,q}^{(n),q}$.

Theorem 2.39.

$$\sum_{s=0}^{\nu} \binom{\nu}{s}_{\frac{1}{q}} \mathrm{B}_{\mathrm{NWA},s}^{(n),q} \mathrm{B}_{\mathrm{NWA},\nu-s}^{(-n),q} = \delta_{\nu,0}. \tag{99}$$

$$\sum_{s=0}^{\nu} \binom{\nu}{s}_{\frac{1}{q}} \mathrm{F}_{\mathrm{NWA},s}^{(n),q} \mathrm{F}_{\mathrm{NWA},\nu-s}^{(-n),q} = \delta_{\nu,0}. \tag{100}$$

Proof. Put $x = y = 0$ in (95) and (96). □

Theorem 2.40. *A q-analogue of* [38, p. 142]. *Assume that $f(x)$ is analytic with q-Taylor expansion*

$$f(x)=\sum_{\nu=0}^{\infty} D_{\frac{1}{q}}^{\nu} f(0)\frac{x^{\nu}}{\{\nu\}_q!}. \tag{101}$$

Then we can express powers of $\triangle_{\text{NWA}}^{q}$ and ∇_{NWA}^{q} operating on $f(x)$ as powers of D_q as follows. These series converge when the absolute value of x is small enough.

$$\underset{\omega_1,\ldots,\omega_n}{(\triangle_{\text{NWA}}^{q})^{n}} f(x)=\sum_{\nu=0}^{\infty} D_{\frac{1}{q}}^{\nu+n} f(0)\frac{\text{B}_{\text{NWA},\nu}^{(-n),q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!}, \tag{102}$$

$$\underset{\omega_1,\ldots,\omega_n}{(\nabla_{\text{NWA}}^{q})^{n}} f(x)=\sum_{\nu=0}^{\infty} D_{\frac{1}{q}}^{\nu} f(0)\frac{\text{F}_{\text{NWA},\nu}^{(-n),q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!}. \tag{103}$$

Proof. Use (26), (17) and (61), (17) respectively. □

Now put $f(x)=\text{E}_{\frac{1}{q}}(xt)$ to obtain the generating function of the pseudo-q-Bernoulli- and pseudo-q-Euler polynomials of negative order.

$$\frac{\prod_{k=1}^{n}(\text{E}_{\frac{1}{q}}(\omega_k t)-1)\text{E}_{\frac{1}{q}}(xt)}{t^n\prod_{k=1}^{n}\omega_k}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\text{B}_{\text{NWA},\nu}^{(-n),q}(x|\omega_1,\ldots,\omega_n), \tag{104}$$

$$\frac{\prod_{k=1}^{n}(\text{E}_{\frac{1}{q}}(\omega_k t)+1)\text{E}_{\frac{1}{q}}(xt)}{2^n}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\text{F}_{\text{NWA},\nu}^{(-n),q}(x|\omega_1,\ldots,\omega_n). \tag{105}$$

The reason for the difference in appearance compared to the original for the following equation is that one of the function arguments is a Ward number. The following two theorems also obtain for β^q polynomials.

Theorem 2.41. *A q-analogue of* [37, p. 191, (10)]*:*

$$\text{B}_{\text{NWA},\nu}^{(m),q}(x\oplus^q \overline{n}_{\frac{1}{q}})=\sum_{k=0}^{\min(\nu,n)}\binom{n}{k}\frac{\{\nu\}_q!}{\{\nu-k\}_q!}\text{B}_{\text{NWA},\nu-k}^{(m-k),q}(x). \tag{106}$$

Proof. Use (13) and (26). □

Theorem 2.42. *A q-analogue of* [33, p. 133, (3)]*:*

$$\frac{\{\nu\}_q!}{\{\nu-n\}_q!}x^{\nu-n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\text{B}_{\text{NWA},\nu}^{(n),q}(x\oplus^q \overline{k}_{\frac{1}{q}}). \tag{107}$$

Proof. Use (14) and (26). □

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$$hu'_x = \varDelta u_x - \tfrac{h}{2}\varDelta u'_x + \frac{B_1 h^2}{1.2}\varDelta u''_x - \frac{B_2 h^4}{1.2.3.4}\varDelta u_x^{\mathrm{IV}} + \text{etc.}$$

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$$hu'_x = \varDelta u_x - \tfrac{h}{2}\varDelta u'_x + \frac{B_1 h^2}{1.2}\varDelta u''_x - \frac{B_2 h^4}{1.2.3.4}\varDelta u_x^{\mathrm{IV}} + \text{etc.}$$

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