# $q$-DEFORMED MATRIX PSEUDO–GROUPS

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Abstract. The quantum groups and Hopf algebras have invaded theoretical physics. We give an alternate (and possibly more understandable) description of these phenomena in this paper. In another paper [9] the author has introduced a $q$-umbral calculus in the spirit of Rota. This umbral calculus contains two dual $q$–additions, the Nalli–Ward–Alsalam (NWA) $q$–addition and the Jackson–Hahn–Cigler (JHC) $q$–addition. In a further paper [10], a matrix calculus was built up from the Jackson $q$-exponential function. It is well-known that Lie groups (one-parameter subgroups) can be made up from the exponential function applied to the Lie algebra (e.g. Pauli matrices). A combination of these two techniques will lead to a new and natural $q$-deformation of matrix Lie groups. We will use two matrix multiplications, the first one is ordinary multiplication, and the second one contains an involution operator $\tau : q \rightarrow \frac{1}{q}$. We supply these two matrix multiplications with an attractive associative structure. It turns out that these deformed matrix pseudo–groups have quite a different character than the well-known quantum groups. In spite of this, we have decided to keep the same notation for these $q$-deformed matrix groups as for the corresponding quantum groups. The first matrix pseudo-group to be defined $SO_q(2)$, is not seen so often explicitly in the quantum group context. However several physicists use the corresponding $SO_q(3)$ symmetric quantum mechanics [11], [30].

## Introduction

The quantum groups and Hopf algebras have invaded theoretical (and experimental) physics. Almost all parts of theoretical physics have received its quantum analogue in the form of $SU_q(2)$, etc. However a comparison with ordinary high school textbooks on the corresponding subjects is hardly possible, the reason is that quantum groups are not easily applicable to ordinary physics experiments. We have decided to introduce a new $q$-deformation of matrix Lie groups in this paper, which will be applicable to quantum systems, as well as to particle physics. This new kind of matrix Lie groups has two multiplications, and these are supplied with an associative structure.

We give a brief overview of the history of quantum groups before turning to a description of the mathematical details of the author's new method.

$q$-Calculus has wideranging applications in theoretical physics and analytic number theory. One example in theoretical physics is the so-called hadronic mechanics. Adrian A. Albert (1905–1972) in [1], introduced a nonassociative algebra together with two products, one a simple composition and the second an antisymmetric product. Here we can see a remote connection between this simple composition and Ward $q$-addition on one hand, and a connection between this antisymmetric product and the Hahn $q$-addition on the other. Anyway, the work of Albert led Santilli in his thesis 1967 [26, (8), p. 573] to investigate a $q$-analogue of the Heisenberg commutator in the form $[A, B] = pAB - qBA$. Santilli, a nuclear physicist, was born in Italy in 1935, and moved to Harvard, USA, in 1978. In the Proceedings of the first workshop on hadronic mechanics Cambridge, Mass., 1983, twenty-six papers appeared.

Later, Santilli moved to Florida, where he founded the Hadronic journal and became head of his own institute.

Then quantum groups were introduced in the mid-eighties by Drinfeld and Jimbo. Quantum groups and their representations today form an important part of both mathematics and theoretical physics. The represențions of quantum groups are the wellknown $q$-special functions. About 1988 S.L. Woronowicz introduced compact matrix quantum groups. Compact matrix quantum groups are abstract structures on which the "continuous functions" on the structure are given by elements of a $C^\star$-algebra. The geometry of a compact matrix quantum group is a special case of a noncommutative geometry. In the same year Manin introduced a general construction for quantum groups as linear transformations on the quantum superplane. This was immediately followed by the introduction of the Wess-Zumino-Witten model, a simple model of conformal field theory whose solutions are realized by affine Kac-Moody algebras.

In 1996 Kadeisvili [21] gave a mathematical description of the Lie-Santilli isotheory, which complements the book Elements of hadronic mechanics.

Finally, in Santilli's latest paper [27, 1.33], the operator expression $[A, B] = AB - qBA$ is encountered, where $A, B$ are noncommutative operators and $q$ is a non-zero scalar.

J. Cigler [3, p.88] has given a mathematical counterpart of this in terms of a number of identities for linear operators on the vector space

$P$ of all polynomials in the variabel $x$ over the field $\mathbb{R}$.

$$D_q\mathbf{x} - q\mathbf{x}D_q = I. \tag{1}$$

$$D_q\mathbf{x} - \mathbf{x}D_q = \epsilon. \tag{2}$$

J. Cigler is still active in Vienna and has a most distinuished school for his students there. It can be said that the research-works by Santilli, the team Wess-Zumino-Witten, and Cigler formed three steps that lead to $q$-calculus.

## 1. First matrix calculations

**Definition 1.** Matrix elements will always be denoted $(i, j)$. Here $i$ denotes the row and $j$ denotes the column. The matrix elements range from 0 to $n-1$. Juxtaposition of matrices will always be interpreted as matrix multiplication. If $A$ and $B$ are commuting matrices of the same dimension, we define $A \oplus_q B$ as a matrix with matrix elements $A(i, j) \oplus_q B(i, j)$. If $A$ and $B$ are commuting matrices of the same dimension, we define $A \boxplus_q B$ as a matrix with matrix elements $A(i, j) \boxplus_q B(i, j)$.

The following object $\mathbb{C}_q^{n,n}$ is an infinite set of $n \times n$ matrices, closed under the two $q$-additions $\oplus_q$ and $\boxplus_q$ and ordinary matrix multiplication $\cdot$, denoted by juxtaposition. A $q$-pseudo-ring $\mathbb{C}_q^{n,n} = (\mathbb{C}_q^{n,n}; \oplus_q, \boxplus_q, \cdot)$ is the set of all matrices $\alpha, \beta, \gamma \dots$ with the property $\alpha\beta \sim \beta\alpha$, together with

$$(\alpha \oplus_q \beta) \oplus_q \gamma \sim \alpha \oplus_q (\beta \oplus_q \gamma), \tag{3}$$

$$\alpha \oplus_q \beta \sim \beta \oplus_q \alpha, \tag{4}$$

$$\alpha \oplus_q \theta \sim \alpha. \tag{5}$$

This $\theta$ is called the zero matrix.

$$\alpha \boxplus_q (-\alpha) \sim \theta, \tag{6}$$

$$(\alpha\beta)\gamma \sim \alpha(\beta\gamma). \tag{7}$$

There is $1 \in \mathbb{C}_q^{n,n}$, such that

$$1\alpha \sim \alpha 1 \sim \alpha \tag{8}$$

The distribitive law holds:

$$\alpha(\beta \oplus_q \gamma) \sim \alpha\beta \oplus_q \alpha\gamma, \tag{9}$$

$$(\beta \oplus_q \gamma)\alpha \sim \beta\alpha \oplus_q \gamma\alpha. \tag{10}$$

Let $\psi$ and $\varphi \in \mathbb{R}_{\oplus_q}$.

We continue with some new terminology. Since we will often use an inversion of the basis we start with this operation in several disguises.

**Definition 2.** The inversion of the basis $q \to \frac{1}{q}$ is denoted $\tau$.

Sometimes we only want to invert factors which depend on a certain variable. In order to do this, we can specify the function argument. The inversion of the basis $q \to \frac{1}{q}$ of functions depending of $x$ is denoted $\tau_x$. So if we have a product or sum of functions, the operator $\tau_x$ only inverts the basis for the members which depend on $x$.

This involution makes orthogonal matrices possible and commutes nicely with complex conjugation when $q$ is real.

**Definition 3.** The $q$-real numbers $\mathbb{R}_{\oplus_q}$ is the set generated by real letters together with the operators $\boxplus_q$, $\oplus_q$. Define an absolute value $|\alpha|$ of an element $\alpha \in \mathbb{R}_{\oplus_q}$ as the absolute value for the case $q = 1$. This means that we just add and subtract the corresponding absolute values without bothering about $q$-additions. We can then define inequalities between these absolute values of elements in $\mathbb{R}_{\oplus_q}$.

**Definition 4.** Let $\psi$ and $\varphi \in \mathbb{R}_{\oplus_q}$. The $q$-Kronecker delta is defined by

$$\delta_{\psi,\varphi} \equiv \begin{cases} 1, & \text{if } \psi \sim \varphi \text{ ;} \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

We will mostly study matrices in this article, these matrices will always be $q$-analogues of objects in $\mathrm{GL}(n, \mathbb{C})$.

**Definition 5.** Denote by $E^\star$ the punctured unit disc, and let $A$ be $Mat(n)^2$. We define the most general matrix pseudo-group, denoted $\mathrm{GL}_q(n, \mathbb{C})$ by

$$\mathrm{GL}_q(n, \mathbb{C}) \equiv \{A : E^\star \to \mathrm{GL}(n, \mathbb{C})\}. \tag{12}$$

$\mathrm{GL}_q(n, \mathbb{C})$ is a $q$-analogue of the group of all linear automorphisms of $\mathbb{C}^n$. Contrary to the case $q = 1$, we will define two matrix multiplications.

**Definition 6.** There are two matrix multiplications: $\cdot$ and $\cdot_q$. The multiplication $\cdot$ is used for ordinary matrix multiplication whereas $\cdot_q$ is defined as follows: Let $\alpha(q)$ and $\beta(q)$ be $n \times n$ matrices in $\mathrm{GL}_q(n, \mathbb{C})$, with matrix elements $\alpha_{ij}$ and $\beta_{ij}$, respectively. Then we define

$$(\alpha \cdot_q \beta)_{ij} \equiv \sum_{m=0}^{n-1} \alpha_{im} \tau(\beta_{mj}). \tag{13}$$

We can modify this product slightly, writing

$$(\alpha \cdot_{x,q} \beta)_{ij} \equiv \sum_{m=0}^{n-1} a_{im} \tau_x(\beta_{mj}). \tag{14}$$

**Definition 7.** Let $\alpha = [a_{ij}]$ be an $m \times n$ matrix with matrix elements $a_{ij}$. The conjugate of $\alpha$ is the $m \times n$ matrix $\overline{\alpha} = [\overline{a_{ij}}]$.

Let $0 < q < 1$. The conjugate transpose of $\alpha$ is the $n \times m$ matrix $\alpha^\star \equiv \tau[\overline{(a_{ij})}]^T$. Here $\tau$ is included in $\star$, so we can use ordinary matrix multiplication.

**Theorem 1.1.** *Properties of the conjugate transpose. Let $\alpha$ and $\beta$ be $n \times n$ matrices and $q$ real. Then*

$$(\alpha^\star)^\star \sim \alpha, \tag{15}$$

$$(\alpha \oplus_q \beta)^\star \sim \alpha^\star \oplus_q \beta^\star, \tag{16}$$

$$(z\alpha)^\star \sim \tau(\overline{z})\alpha^\star, \text{ for any scalar } z \in \mathbb{C}, \tag{17}$$

$$(\alpha \cdot \beta)^\star = \beta^\star \cdot \alpha^\star \tag{18}$$

*Proof.* Formula (17) is proved as follows.

$$((z\alpha)^\star)_{ij} \sim \left(\tau(\overline{z\alpha})^T\right)_{ij} \sim \tau\overline{z} \cdot \left(\tau(\overline{\alpha})^T\right)_{ij} \sim \tau\overline{z}\,(\alpha^\star)_{ij}\,. \tag{19}$$

Formula (18) is proved as follows.

$$((\alpha \cdot \beta)^\star)_{ij} = \left(\sum_{m=0}^{n-1} \alpha_{im}\beta_{mj}\right)^\star = \tau\left(\sum_{m=0}^{n-1} \overline{\beta}_{jm}\overline{\alpha}_{mi}\right) \quad \square$$
$$= \left((\tau\overline{\beta})^T \cdot (\tau\overline{\alpha})^T\right)_{ij} = ((\beta^\star \cdot \alpha^\star))_{ij}\,.$$

**Definition 8.** A square matrix pseudo-group $H$ is called Hermitian if $H^\star = H$.

**Definition 9.** The $q$-determinant of a $2 \times 2$ matrix

$$\alpha \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{21}$$

is defined by

$$det_q\alpha \equiv a_{11}\tau(a_{22}) - a_{12}\tau(a_{21}). \tag{22}$$

The $q$-determinant of an $n \times n$ matrix

$$\alpha \equiv \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{23}$$

is defined by

$$det_q M \equiv \sum_{\pi \in S_n} sign\pi a_{\pi(1)1}\tau(a_{\pi(2)2})a_{\pi(3)3}\tau(a_{\pi(4)4})\dots a_{\pi(n)n}, \tag{24}$$

$n$ odd. If $n$ is even, the last factor is $\tau(a_{\pi(n)n})$.

We can define several $q$-scalar products.

**Definition 10.** The Euclidean $q$-scalar product of two vectors $e_1$ and $e_2$, each of dimension $n$, with matrix elements $e_{1k}$ and $e_{2k}$ is defined by the following mapping $Mat(n)^2 \to \mathbb{R}$:

$$e_1 \cdot_q e_2 \equiv \sum_{k=1}^{n} e_{1k}\tau(e_{2k}). \tag{25}$$

The pseudo-group $O_q(n) = \{O : O_\psi \cdot_q O_\varphi = \delta_{\psi,\varphi}\}$. Here $O_\psi$ and $O_\varphi$ are either certain rows or certain columns in $O_q(n)$. The elements in $O_q(n)$ are called $q$-orthogonal matrices.

The definite hermitian $q$-scalar product of two $n$-dimensional vectors $e_1$ and $e_2$, with matrix elements $e_{1k}$ and $e_{2k}$ in $\mathbb{C}^n$ is defined by

$$e_1 \cdot_q e_2 \equiv \sum_{k=1}^{n} e_{1k}\overline{\tau(e_{2k})}. \tag{26}$$

The pseudo-group $U_q(n) = \{U : U_\psi \cdot_q U_\varphi = \delta_{\psi,\varphi}\}$. Here $U_\psi$ and $U_\varphi$ are either certain rows or certain columns in $U_q(n)$. We call these matices $q$-orthogonal. An equivalent definition is

$$A^{-1} = A^\star. \tag{27}$$

Here $\tau$ is included in $\star$, so we can use the ordinary matrix multiplication. The corresponding norm of a vector $e_1$ in $\mathrm{GL}_q(n, \mathbb{C})$ is defined by

$$\|e_1\| \equiv \sqrt{e_1 \cdot_q e_1}. \tag{28}$$

For vectors with hermitian $q$-scalar product we have the $q$-Cauchy-Schwarz inequality:

$$(e_1 \cdot_q e_2)\tau(\overline{e_1 \cdot_q e_2}) \le \|e_1\|^2\|e_2\|^2. \tag{29}$$

This inequality is justified by the following computation:

$$\begin{aligned} 0 &\le (re_1 + te_2) \cdot_q (re_1 + te_2) = r^2 e_1 \cdot_q e_1 + tre_2 \cdot_q e_1 + rte_1 \cdot_q e_2 + t^2 e_2 \cdot_q e_2 \\ &= r^2\|e_1\|^2 + rt(e_1 \cdot_q e_2 + \tau(\overline{e_1 \cdot_q e_2})) + t^2\|e_2\|^2. \end{aligned} \tag{30}$$

Now put $r = \|e_2\|$ and $t = -\|e_2\|^{-1}(e_1 \cdot_q e_2)$ to obtain

$$\|e_1\|^2\|e_2\|^2 - [(e_1 \cdot_q e_2)^2 + (e_1 \cdot_q e_2)\tau(\overline{e_1 \cdot_q e_2})] + (e_1 \cdot_q e_2)^2 \ge 0. \tag{31}$$

Finally we obtain

$$\|e_1\|^2\|e_2\|^2 \ge (e_1 \cdot_q e_2)\tau(\overline{e_1 \cdot_q e_2}). \tag{32}$$

There is also an indefinite $q$-scalar product of two $n$-dimensional vectors $e_1$ and $e_2$, with matrix elements $e_{1k}$ and $e_{2k}$ defined by

$$e_1 \cdot_q e_2 \equiv \sum_{k=1}^{l} e_{1k}\tau(e_{2k}) - \sum_{k=l+1}^{n} e_{1k}\tau(e_{2k}). \tag{33}$$

The indefinite hermitian $q$-scalar product of two $n$ vectors $e_1$ and $e_2$, with matrix elements $e_{1k}$ and $e_{2k}$ in $\mathbb{C}^n$ is defined by the following mapping $Mat(n)^2 \to \mathbb{R}$:

$$e_1 \cdot_q e_2 \equiv \sum_{k=1}^{l} e_{1k}\overline{\tau(e_{2k})} - \sum_{k=l+1}^{n} e_{1k}\overline{\tau(e_{2k})}. \tag{34}$$

1.1. **Functions of matrices.** In this subsection we treat $q$-matrices in a more general way. The concept of formal power series is introduced. The matrix norm is defined as for the case $q = 1$. The two $q$-exponential functions also play an important role, and some of the concepts from the previous subsection can be used in this context. The addition formula (48) is the general form for the first matrix multiplication $\cdot$ in $SO_q(2)$ and $SU_q(2)$. The addition formula (49) is the general form for the second matrix multiplication $\cdot_q$ in $SO_q(2)$ and $SU_q(2)$.

**Definition 11.** Let $A$ be a quadratic matrix. If $f(A)$ is analytic in a vicinity of its complex eigenvalues, we put

$$f(A) \equiv \sum_{k=0}^{\infty} a_k A^k. \tag{35}$$

We define the norm of a quadratic matrix $A$ by

$$||A|| \equiv \sup_n\{x \in \mathbb{C},\ x \neq 0\} \frac{|Ax|}{|x|}. \tag{36}$$

**Lemma 1.2.** *Properties of the matrix norm.*

$$|Ax| \leq ||A|| \cdot |x|, \tag{37}$$

$$|A_{ij}| \leq ||A||, \tag{38}$$

$$||A|| \leq \left( \sum_{i,j=0}^{n-1} |A_{ij}|^2 \right)^{\frac{1}{2}}, \tag{39}$$

$$||AB|| \leq ||A|| \cdot ||B||, \tag{40}$$

$$||tA|| = |t| \cdot ||A||, t \in \mathbb{C}. \tag{41}$$

We infer that the series (35) is convergent if the function $f$ is entire.

**Definition 12.** The $q$-derivate of a matrix is defined as the matrix with matrix elements equal to the $q$-derivate of the original matrix elements.

**Theorem 1.3.**

$$\mathrm{D}_q AB = \mathrm{D}_q(A)\epsilon B + A\mathrm{D}_q B. \tag{42}$$

*Proof.* We show that the matrix elements are equal.

$$\begin{aligned} &\mathrm{D}_q(AB)_{ij} = \sum_{m=0}^{n-1} \mathrm{D}_q(A_{im}B_{mj}) = \\ &\sum_{m=0}^{n-1} \mathrm{D}_q(A_{im})\epsilon B_{mj} + (A_{im}\mathrm{D}_q B_{mj} = (\mathrm{D}_q(A)\epsilon B + A\mathrm{D}_q B)_{ij}. \end{aligned} \tag{43}$$

□

**Definition 13.** The $q$-derivate of (35) with respect to the matrix $A$ is defined as

$$\sum_{k=0}^{\infty} \{k+1\}_q a_{k+1} A^k. \tag{44}$$

**Definition 14.** Let $A$ be an $n \times n$ matrix, $0 < |q| < 1$ and $||A|| < |1-q|^{-1}$. Then

$$\mathrm{E}_q(A) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} A^k. \tag{45}$$

**Theorem 1.4.** *Let $I$ denote a unit matrix. The meromorphic continuation of $\mathrm{E}_q(A)$ is*

$$F(A) \equiv (\prod_{m=0}^{\infty} I - q^m(1-q)A)^{-1}. \tag{46}$$

We can say that the matrix function $F(A)$ has simple poles at $A = I\frac{q^{-k}}{1-q}$, $k \in \mathbb{N}$. $F(A)$ is a good substitute for $E_q(A)$ in the whole complex plane. We shall however continue to call this function $E_q(A)$, since it plays an important role in the operator theory.

**Definition 15.** If $A$ is an $n \times n$ matrix, then we may write

$$\mathrm{E}_{\frac{1}{q}}(A) \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{\{k\}_q!} A^k. \tag{47}$$

**Theorem 1.5.**

$$\mathrm{E}_q(A \oplus_q B) = \mathrm{E}_q(A)\mathrm{E}_q(B). \tag{48}$$

$$\mathrm{E}_q(A \boxplus_q B) = \mathrm{E}_q(A)\mathrm{E}_{\frac{1}{q}}(B). \tag{49}$$

$$\mathrm{D}_{q,x}\mathrm{E}_q(Ax) = A\mathrm{E}_q(Ax). \tag{50}$$

If $A$ is diagonalizable, meaning that $A = SBS^{-1}$, where $B$ is a diagonal matrix, and $S$ invertible, we can compute $\mathrm{E}_q(A)$ as

$$\mathrm{E}_q(A) = S\mathrm{E}_q(B)S^{-1}. \tag{51}$$

**Theorem 1.6.** *Let the eigenvalues of the quadratic $n \times n$ matrix $A$ be $\{\lambda_i\}_{i=1}^k$ with multiplicities $\{n_i\}_{i=1}^k$, $\sum_{i=1}^k n_i = n$. According to the Cayley-Hamilton theorem, we can compute $\mathrm{E}_q(tA)$ as*

$$\mathrm{E}_q(tA) = \sum_{i=1}^{k} P_i(A)\mathrm{E}_q(t\lambda_i), \tag{52}$$

*where*

$$P_i(\lambda_j) = \delta_{ij}. \tag{53}$$

*Each element in $\mathrm{E}_q(tA)$ is a linear combination of terms of the form $t^j\mathrm{E}_q(t\lambda)$, where $\lambda$ is an eigenvalue of $A$ and $j$ is less than the multiplicity of the eigenvalue $\lambda$. As an example, assume that the multiplicity of $\lambda_i$ is $n_i$. Then we have that*

$$\mathrm{D}_q^k P_i(\lambda_i) = t^k\mathrm{E}_q(t\lambda_i),\ k < n_i. \tag{54}$$

If a given matrix has been transformed to Jordan form $J = \Lambda + N$, with commuting terms $\Lambda$ (diagonal) and $N$ (nilpotent), and then computing $\mathrm{E}_q(\Lambda)$ and $\mathrm{E}_q(N)$, where the second series terminates, formula (48) gives a way of computing $\mathrm{E}_q(\Lambda \oplus_q N)$.

## 2. Compact matrix pseudo-groups

**Definition 16.** A matrix pseudo-group $A_q$ is called compact if

(1) the matrix elements of all matrices in $A_q$ are bounded functions.
(2) The set of matrices in $A_q$ are closed under the two operations . and $\cdot_q$.

We illustrate the general technique on a specific $q$-deformed matrix group, $SO_q(2)$. A general element of $SO_q(2)$ is denoted $O_\psi$, where $\psi \in \mathbb{R}_{\oplus_q}$. We can write $O_\psi$ as follows:

$$O_\psi = \begin{pmatrix} \mathrm{Cos}_q(\psi) & -\mathrm{Sin}_q(\psi) \\ \mathrm{Sin}_q(\psi) & \mathrm{Cos}_q(\psi) \end{pmatrix}. \tag{55}$$

An immediate calculation shows that $det_q O_\psi = 1$. The rows and columns of $O_\psi$ are also orthogonal according to the $q$-scalar product (25). This motivates the notation $SO_q(2)$.

We are now in the position to introduce matrix multiplication in $SO_q(2)$. For $q = 1$, there is only one matrix multiplication. For general

$q$ there should possibly be 2 according to the quantum group concept. If we are lucky, these could correspond to NWA and JHC $q$–addition. We introduce the following two multiplications in $SO_q(2)$, the first one is commutative.

**Definition 17.** Ordinary matrix multiplication.

$$O_\psi \cdot O_\varphi. \tag{56}$$

$q$-matrix multiplication, denoted $\cdot_q$

$$O_\psi \cdot_q O_\varphi \equiv \begin{pmatrix} \mathrm{Cos}_q(\psi) & -\mathrm{Sin}_q(\psi) \\ \mathrm{Sin}_q(\psi) & \mathrm{Cos}_q(\psi) \end{pmatrix} \begin{pmatrix} \mathrm{Cos}_{\frac{1}{q}}(\varphi) & -\mathrm{Sin}_{\frac{1}{q}}(\varphi) \\ \mathrm{Sin}_{\frac{1}{q}}(\varphi) & \mathrm{Cos}_{\frac{1}{q}}(\varphi) \end{pmatrix}. \tag{57}$$

By the Jackson [18] addition theorems for $q$-trigonometric functions we find

**Theorem 2.1.**

$$\begin{pmatrix} \mathrm{Cos}_q(\psi) & -\mathrm{Sin}_q(\psi) \\ \mathrm{Sin}_q(\psi) & \mathrm{Cos}_q(\psi) \end{pmatrix} \begin{pmatrix} \mathrm{Cos}_q(\varphi) & -\mathrm{Sin}_q(\varphi) \\ \mathrm{Sin}_q(\varphi) & \mathrm{Cos}_q(\varphi) \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}_q(\psi \oplus_q \varphi) & -\mathrm{Sin}_q(\psi \oplus_q \varphi) \\ \\ \mathrm{Sin}_q(\psi \oplus_q \varphi) & \mathrm{Cos}_q(\psi \oplus_q \varphi) \\ \\ \end{pmatrix}. \tag{58}$$

$$\begin{pmatrix} \mathrm{Cos}_q(\psi) & -\mathrm{Sin}_q(\psi) \\ \mathrm{Sin}_q(\psi) & \mathrm{Cos}_q(\psi) \end{pmatrix} \begin{pmatrix} \mathrm{Cos}_{\frac{1}{q}}(\varphi) & -\mathrm{Sin}_{\frac{1}{q}}(\varphi) \\ \mathrm{Sin}_{\frac{1}{q}}(\varphi) & \mathrm{Cos}_{\frac{1}{q}}(\varphi) \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}_q(\psi \boxplus_q \varphi) & -\mathrm{Sin}_q(\psi \boxplus_q \varphi) \\ \mathrm{Sin}_q(\psi \boxplus_q \varphi) & \mathrm{Cos}_q(\psi \boxplus_q \varphi) \end{pmatrix}. \tag{59}$$

This implies

**Theorem 2.2.** *The matrix pseudo-group*

$$(SO_q(2), \cdot, \cdot_q) \tag{60}$$

*is closed under the two operations. The function argument, including the zero $\theta$, belongs to the alphabet. The unit of $SO_q(2)$ is the unit matrix, corresponding to $\theta$. The first operation $\cdot$ is commutative. The operations $\cdot$ and $\cdot_q$ correspond exactly to NWA and JHC $q$–addition. Only the second operation $\cdot_q$ admits an inverse $-\psi$ of an arbitrary element $\psi$.*

The Lie algebra can be computed in the usual way:

$$D_{q,\psi}O_\psi = \begin{pmatrix} \mathrm{Cos}_q(\psi) & -\mathrm{Sin}_q(\psi) \\ \mathrm{Sin}_q(\psi) & \mathrm{Cos}_q(\psi) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{61}$$

The last matrix is the Lie algebra, just as in the undeformed case.The set of improper rotations is

$$\begin{pmatrix} \mathrm{Cos}_q(\psi) & \mathrm{Sin}_q(\psi) \\ \mathrm{Sin}_q(\psi) & -\mathrm{Cos}_q(\psi) \end{pmatrix}, \psi \in \mathbb{R}_{\oplus_q}. \tag{62}$$

This corresponds to a reflection about a line through the origin.

We now turn to $SU_q(2)$.

**Definition 18.** The general form of matrices in $SU_q(2)$ is for $\psi, \phi, \alpha \in \mathbb{R}_{\oplus_q}$.

$$U_{\psi,\phi,\alpha} \equiv \begin{pmatrix} \mathrm{Cos}_q(\psi)\mathrm{E}_q(i\phi) & -\mathrm{Sin}_q(\psi)\mathrm{E}_q(i\alpha) \\ \mathrm{Sin}_q(\psi)\mathrm{E}_q(-i\alpha) & \mathrm{Cos}_q(\psi)\mathrm{E}_q(-i\phi) \end{pmatrix}, \tag{63}$$

with $q$-determinant 1.

**Theorem 2.3.** *Associativity for $SU_q(2)$.*

$$(U_{\psi_1,\phi_1,\alpha_1} \cdot U_{\psi_2,\phi_2,\alpha_2}) \cdot_q U_{\psi_3,\phi_3,\alpha_3} = U_{\psi_1,\phi_1,\alpha_1} \cdot (U_{\psi_2,\phi_2,\alpha_2} \cdot_q U_{\psi_3,\phi_3,\alpha_3}). \tag{64}$$

*We have chosen a special combination of multiplications, however the following proof will show that the formula holds for any such combination.*

*Proof.* For brevity, we only compare the matrix indices $A_{11}$. The LHS of (64) has the form

$$\left[\begin{pmatrix} \mathrm{Cos}_q(\psi_1)\mathrm{E}_q(i\phi_1) & -\mathrm{Sin}_q(\psi_1)\mathrm{E}_q(i\alpha_1) \\ \mathrm{Sin}_q(\psi_1)\mathrm{E}_q(-i\alpha_1) & \mathrm{Cos}_q(\psi_1)\mathrm{E}_q(-i\phi_1) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Cos}_q(\psi_2)\mathrm{E}_q(i\phi_2) & -\mathrm{Sin}_q(\psi_2)\mathrm{E}_q(i\alpha_2) \\ \mathrm{Sin}_q(\psi_2)\mathrm{E}_q(-i\alpha_2) & \mathrm{Cos}_q(\psi_2)\mathrm{E}_q(-i\phi_2) \end{pmatrix}\right]$$
$$\cdot_q \begin{pmatrix} \mathrm{Cos}_q(\psi_3)\mathrm{E}_q(i\phi_3) & -\mathrm{Sin}_q(\psi_3)\mathrm{E}_q(i\alpha_3) \\ \mathrm{Sin}_q(\psi_3)\mathrm{E}_q(-i\alpha_3) & \mathrm{Cos}_q(\psi_3)\mathrm{E}_q(-i\phi_3) \end{pmatrix}. \tag{65}$$

The first index $A_{11}$ is

$$[\mathrm{Cos}_q(\psi_1)\mathrm{Cos}_q(\psi_2)\mathrm{E}_q(i(\phi_1 \oplus_q \phi_2)) - \mathrm{Sin}_q(\psi_1)\mathrm{Sin}_q(\psi_2)\mathrm{E}_q(i(\alpha_1 \ominus_q \alpha_2))]\,\mathrm{Cos}_{\frac{1}{q}}(\psi_3)\mathrm{E}_{\frac{1}{q}}(i(\phi_3) -$$
$$[\mathrm{Cos}_q(\psi_1)\mathrm{Sin}_q(\psi_2)\mathrm{E}_q(i(\phi_1 \oplus_q \alpha_2)) + \mathrm{Sin}_q(\psi_1)\mathrm{Cos}_q(\psi_2)\mathrm{E}_q(i(\alpha_1 \ominus_q \phi_2))]\,\mathrm{Sin}_{\frac{1}{q}}(\psi_3)\mathrm{E}_{\frac{1}{q}}(-i(\alpha_3). \tag{66}$$

The second index $A_{11}$ is

$$\mathrm{Cos}_q(\psi_1)\mathrm{E}_q(i\phi_1)\left[\mathrm{Cos}_q(\psi_2)\mathrm{Cos}_{\frac{1}{q}}(\psi_3)\mathrm{E}_q(i(\phi_2 \boxplus_q \phi_3)) - \mathrm{Sin}_q(\psi_2)\mathrm{Sin}_{\frac{1}{q}}(\psi_3)\mathrm{E}_q(i(\alpha_2 \boxminus_q \alpha_3))\right] - \mathrm{Sin}_q(\psi_1)\mathrm{E}_q(i(\alpha_1)\left[\mathrm{Sin}_q(\psi_2)\mathrm{Cos}_{\frac{1}{q}}(\psi_3)\mathrm{E}_q(i(-\alpha_2 \boxplus_q \phi_3)) + \mathrm{Cos}_q(\psi_2)\mathrm{Sin}_{\frac{1}{q}}(\psi_3)\mathrm{E}_q(i(-\phi_2 \boxminus_q \alpha_3))\right]. \tag{67}$$

The expressions are equal, and the same for the three other matrix indices. □

**Definition 19.** To form the inverse of (63), with multiplication $\cdot_q$, we put

$$U^{-1}_{\psi_1,\phi_1,\alpha_1} \equiv U_{\psi_2,\phi_2,\alpha_2}, \tag{68}$$

with the following three conditions on the umbrae:

$$\alpha_1 \sim \alpha_2,\ \phi_1 \sim -\phi_2,\ \psi_1 \sim -\psi_2. \tag{69}$$

Sometimes we need to consider a maximal torus of the form

$$U_\phi \equiv \begin{pmatrix} \mathrm{E}_q(i\phi) & 0 \\ 0 & \mathrm{E}_q(-i\phi) \end{pmatrix}, \tag{70}$$

corresponding to $\psi = 0$ in (63). We introduce the following two multiplications in the maximal torus of $SU_q(2)$, the first one is commutative.

**Theorem 2.4.** *We have the following expression for $U_\phi \cdot U_\varphi$:*

$$\begin{pmatrix} \mathrm{E}_q(i\psi) & 0 \\ 0 & \mathrm{E}_q(-i\psi) \end{pmatrix}\begin{pmatrix} \mathrm{E}_q(i\varphi) & 0 \\ 0 & \mathrm{E}_q(-i\varphi) \end{pmatrix} = \begin{pmatrix} \mathrm{E}_q(i(\psi \oplus_q \varphi)) & 0 \\ 0 & \mathrm{E}_q(-i(\psi \oplus_q \varphi)) \end{pmatrix}. \tag{71}$$

*We have the following expression for $U_\psi \cdot_q U_\varphi$:*

$$\begin{pmatrix} \mathrm{E}_q(i\psi) & 0 \\ 0 & \mathrm{E}_q(-i\psi) \end{pmatrix}\begin{pmatrix} \mathrm{E}_{\frac{1}{q}}(i\varphi) & 0 \\ 0 & \mathrm{E}_{\frac{1}{q}}(-i\varphi) \end{pmatrix} = \begin{pmatrix} \mathrm{E}_q(i(\psi \boxplus_q \varphi)) & 0 \\ 0 & \mathrm{E}_q(-i(\psi \boxplus_q \varphi)) \end{pmatrix}. \tag{72}$$

*Proof.* Use the Jackson [18] addition theorem. □

Then we have

**Theorem 2.5.** *The matrix pseudo-group*

$$(SU_q(2), \cdot, \cdot_q), \tag{73}$$

*in the form of a maximal torus (70), is closed under the two operations. The function argument, including the zero, belongs to the alphabet. The* *first operation $\cdot$ is commutative and associative. The operations $\cdot$ and $\cdot_q$ correspond exactly to NWA and JHC q–addition. Only the second operation admits an inverse. The corresponding hermitian q-scalar product is (26).*

In this paper we are going to use several concepts from Lie group theory in the spirit of Curtis [4]. In most cases the generalization to pseudo-groups is quite natural.

The maximal torus in $U_q(n)$ has the form

$$\begin{pmatrix} \mathrm{E}_q(i\phi_1) & \dots & \dots & 0 \\ 0 & \mathrm{E}_q(i\phi_2) & \dots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \dots & \dots & \mathrm{E}_q(i\phi_n) \end{pmatrix}. \tag{74}$$

**Definition 20.** An action of a matrix pseudo-group $G_q$ on itself consists of a left translation $L_g : G_q \to G_q$ defined by $h \to g \cdot h$, and a right translation $R_g : G_q \to G_q$ defined by $h \to h \cdot_q \tau(g^{-1})$.

The conjugation $I_g(h) : G_q \to G_q$ is given by $h \to g \cdot h \cdot_q \tau(g^{-1})$.

**Definition 21.** The center of a matrix pseudo-group $G_q$ is the set of all $g \in G_q$ such that $I_g(h) = h,\ h \in G_q$.

**Theorem 2.6.** *The center of $U_q(n)$ is $\mathrm{E}_q(i\alpha)I_n$, where $I_n$ denotes the unit $n \times n$ matrix.*

*The center of $SO_q(2n+1)$ is $I_{2n+1}$.*

*The center of $SO_q(2n)$ is $\pm I_{2n}$.*

The maximal torus in $SU_q(n)$ has the form

$$\begin{pmatrix} \mathrm{E}_q(i\phi_1) & \dots & \dots & 0 \\ 0 & \mathrm{E}_q(i\phi_2) & \dots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \dots & \dots & \mathrm{E}_q(i\phi_n) \end{pmatrix}, \tag{75}$$

where

$$\phi_1 \boxplus_q \phi_2 \oplus_q \phi_3 \boxplus_q \phi_4 \oplus_q \dots \phi_n \sim \theta. \tag{76}$$

The corresponding $q$-Cartan matrix has the form

$$\begin{pmatrix} i\phi_1 & \dots & \dots & 0 \\ 0 & i\phi_2 & \dots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \dots & \dots & i\phi_n \end{pmatrix}, \tag{77}$$

We can again compute the Lie algebra by operating with $D_{q,\phi}$ on (63).

$$D_{q,\phi}U_{\psi,\phi} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} U_{\psi,\phi}. \tag{78}$$

The first matrix on the right belongs to the Lie algebra of $SU_q(2)$.

A general element $O_\psi \in SO_q(2)$ can be mapped to $SU_q(2)$ in the following two ways.

**Definition 22.** Two $q$-morphisms $SO_q(2) \to SU_q(2)$. We denote the following mappings $F$ and $F_q$:

$$U_\alpha \cdot O_\psi \cdot U_\beta = \begin{pmatrix} \mathrm{Cos}_q(\psi)\mathrm{E}_q(i(\alpha \oplus_q \beta)) & -\mathrm{Sin}_q(\psi)\mathrm{E}_q(i(\alpha \ominus_q \beta)) \\ \mathrm{Sin}_q(\psi)\mathrm{E}_q(-i(\alpha \ominus_q \beta)) & \mathrm{Cos}_q(\psi)\mathrm{E}_q(-i(\alpha \oplus_q \beta)) \end{pmatrix}. \tag{79}$$

$$U_\alpha \cdot O_\psi \cdot_q U_\beta = \begin{pmatrix} \mathrm{Cos}_q(\psi)\mathrm{E}_q(i(\alpha \boxplus_q \beta)) & -\mathrm{Sin}_q(\psi)\mathrm{E}_q(i(\alpha \boxminus_q \beta)) \\ \mathrm{Sin}_q(\psi)\mathrm{E}_q(-i(\alpha \boxminus_q \beta)) & \mathrm{Cos}_q(\psi)\mathrm{E}_q(-i(\alpha \boxplus_q \beta)) \end{pmatrix}. \tag{80}$$

The letters $(\alpha,\ \psi,\ \beta)$ correspond to Euler's angels. The two letters $\alpha,\ \psi$ correspond to spherical coordinates.

**Theorem 2.7.** *The $q$-morphism (80) has an inverse $F_q^{-1}$ defined by the following formula, compare* [31, p. 289].

$$U_\beta \cdot \begin{pmatrix} \mathrm{Cos}_q(\psi)\mathrm{E}_q(i(\alpha \boxminus_q \beta)) & -\mathrm{Sin}_q(\psi)\mathrm{E}_q(i(\alpha \boxplus_q \beta)) \\ \mathrm{Sin}_q(\psi)\mathrm{E}_q(-i(\alpha \boxplus_q \beta)) & \mathrm{Cos}_q(\psi)\mathrm{E}_q(-i(\alpha \boxminus_q \beta)) \end{pmatrix} \cdot_q U_{-\alpha} = O_\psi. \tag{81}$$

**Theorem 2.8.** *The kernel of the $q$-morphism $F_q^{-1}$ is*

$$Ker(F_q^{-1}) = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{Cos}_q(\xi(q)) & 0 \\ 0 & \mathrm{Cos}_q(\xi(q)) \end{pmatrix}\} \tag{82}$$

This means that we can say as in the ordinary case, that

$$SU_q(2) \cong SO_q(2) \times \mathbb{Z}_2. \tag{83}$$

**Theorem 2.9.** *A $q$-morphism theorem.*

$$(U_\alpha O_{\phi \boxplus_q \psi}) \cdot_q U_\beta = U_{\alpha \oplus_q \gamma, \phi} \cdot_q U_{-\gamma \oplus_q \beta, \psi}. \tag{84}$$

We are going to construct irreducible representations of $SU_q(2)$ in the following way, a $q$-analogue of [28, p. 112].

Start with

$$U_{\psi,\phi,\alpha} \equiv \begin{pmatrix} \mathrm{Cos}_q(\psi)\mathrm{E}_q(i\phi) & -\mathrm{Sin}_q(\psi)\mathrm{E}_q(i\alpha) \\ \mathrm{Sin}_q(\psi)\mathrm{E}_q(-i\alpha) & \mathrm{Cos}_q(\psi)\mathrm{E}_q(-i\phi) \end{pmatrix}. \tag{85}$$

Then

$$U^{-1}_{\psi,\phi,\alpha} = \begin{pmatrix} \mathrm{Cos}_q(\psi)\mathrm{E}_q(-i\phi) & \mathrm{Sin}_q(\psi)\mathrm{E}_q(i\alpha) \\ -\mathrm{Sin}_q(\psi)\mathrm{E}_q(-i\alpha) & \mathrm{Cos}_q(\psi)\mathrm{E}_q(i\phi) \end{pmatrix}. \tag{86}$$

Consider a vector

$$\vec{z} \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

and the set of functions

$$e^n(\vec{z}) \equiv \frac{z_1^{j+n} z_2^{j-n}}{\sqrt{(\{j+n\}_q!\{j-n\}_q!)}}. \tag{87}$$

Here $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The space of homogeneous polynomials of degree $2j$ is spanned by the functions $e^m, m = -j, -j+1, \ldots, j$. These representations will be indexed by the values $-j, -j+1 \ldots$.

We are going to use a special matrix multiplication defined as follows:

**Definition 23.**

$$U^{-1}_{\psi,\phi,\alpha}\vec{z} \equiv \begin{pmatrix} \mathrm{Cos}_q(\psi)\mathrm{E}_q(-i\phi)z_1 \oplus_q Sin_q(\psi)\mathrm{E}_q(i\alpha)z_2 \\ -Sin_q(\psi)\mathrm{E}_q(-i\alpha)z_1 \oplus_q \mathrm{Cos}_q(\psi)\mathrm{E}_q(i\phi)z_2 \end{pmatrix} \tag{88}$$

This implies that

$$\begin{aligned} e^n(U^{-1}_{\psi,\phi,\alpha}\vec{z}) &= \frac{1}{\sqrt{(\{j+n\}_q!\{j-n\}_q!)}} \big((\mathrm{Cos}_q(\psi)\mathrm{E}_q(-i\phi)z_1 \oplus_q Sin_q(\psi)\mathrm{E}_q(i\alpha)z_2)^{j+n} \\ &(-Sin_q(\psi)\mathrm{E}_q(-i\alpha)z_1 \oplus_q \mathrm{Cos}_q(\psi)\mathrm{E}_q(i\phi)z_2)^{j-n}\big) = \frac{1}{\sqrt{(\{j+n\}_q!\{j-n\}_q!)}} \\ &\sum_{s=0}^{j+n}\sum_{t=0}^{j-n} \binom{j+n}{s}_q \binom{j-n}{t}_q (\mathrm{Cos}_q(\psi)\mathrm{E}_q(-i\phi)z_1)^s (Sin_q(\psi)\mathrm{E}_q(i\alpha)z_2)^{j+n-s} \\ &(-Sin_q(\psi)\mathrm{E}_q(-i\alpha)z_1)^t (\mathrm{Cos}_q(\psi)\mathrm{E}_q(i\phi)z_2)^{j-n-t} = \frac{1}{\sqrt{(\{j+n\}_q!\{j-n\}_q!)}} \\ &\sum_{s=0}^{j+n}\sum_{t=0}^{j-n} \binom{j+n}{s}_q \binom{j-n}{t}_q (-1)^t (\mathrm{Cos}_q(\psi))^{j-n+s-t}) (Sin_q(\psi))^{j+n+t-s} \\ &\mathrm{E}_q(i\phi((\overline{j-n-t}_q) \ominus_q (\overline{s}_q)))\mathrm{E}_q(i\alpha((\overline{j+n-s}_q) \ominus_q (\overline{t}_q)))z_1)^{s+t} z_2^{2j-s-t}. \end{aligned} \tag{89}$$

The representation is accordingly achieved by putting $s = j + m - t$:

$$D^j(U_{\psi,\phi,\alpha})_{mn} = \frac{1}{\sqrt{(\{j+n\}_q!\{j-n\}_q!)}}$$
$$\sum_{t=0}^{j-n} \binom{j+n}{j+m-t}_q \binom{j-n}{t}_q (-1)^t (\mathrm{Cos}_q(\psi))^{2j-n+m-2t} (\mathrm{Sin}_q(\psi))^{-m+n+2t}$$
$$\mathrm{E}_q(i\phi((\overline{j-n-t}_q) \ominus_q (\overline{j+m-t}_q)))\mathrm{E}_q(i\alpha((\overline{t+n-m}_q) \ominus_q (\overline{t}_q)))z_1^{j+m} z_2^{j-m}. \tag{90}$$

The $D^j$ for integer $j$ constitute all the irreducible representations of $SO_q(3)$. We want to remark that there exist several different forms of representations of $SU_q(2)$ in the literature, one of these is due to Finkelstein [12, p. 2633, 4.1].

## References


[1] A.A. Albert, On Jordan algebras of linear transformations. *Trans. Amer. Math. Soc.* **59**, (1946) 524–555.

[2] W.A. Al-Salam, $q$-Bernoulli Numbers and Polynomials, *Math. Nachr.* **17** (1959) 239–260.

[3] J. Cigler, Operatormethoden für $q$-Identitäten. Monatshefte für Mathematik 88,(1979), 87-105.

[4] M. Curtis, Matrix groups. Second edition. Universitext. Springer-Verlag, New York, 1984

[5] T. Ernst, *The history of $q$-calculus and a new method,* Uppsala , 2000.

[6] T. Ernst, A method for $q$-calculus. *J. nonlinear Math. Physics* **10** No.4 (2003), 487-525.

[7] T. Ernst, Some results for $q$-functions of many variables. *Rendiconti di Padova*,**112** (2004), 199-235.

[8] T. Ernst, Some new formulas involving $\Gamma_q$ functions, *Rendiconti di Padova*,**118** (2007), 159–188.

[9] T. Ernst, $q$-complex numbers, a natural consequence of umbral calculus. Submitted.

[10] T.Ernst, $q$-Pascal and $q$-Bernoulli matrices, an umbral approach. Submitted.

[11] J. Feigenbaum, P. Freund, A $q$-deformation of the Coulomb problem. J. Math. Phys. 37 (1996), no. 4, 1602–1616.

[12] R.J.Finkelstein, The $q$-Coulomb problem. J. Math. Phys. 37 (1996), no. 6, 2628–2636.

[13] G.Gasper & M.Rahman, Basic hypergeometric series. Cambridge 1990.

[14] C.F.Gauss, Summatio quarundam serierum singularium. Opera, vol. 2,(1876) pp.16-17.

[15] W. Hahn Beiträge zur Theorie der Heineschen Reihen. Mathematische Nachrichten 2 (1949),340-379.

[16] E.Heine, Über die Reihe $1+[(1-q^\alpha)(1-q^\beta)]/[(1-q)(1-q^\gamma)]x+[(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})]/[(1-q)(1-q^2)(1-q^\gamma(1-q^{\gamma+1})]x^2+\dots$ *J. reine angew. Math.* **32**, (*1846*), 210–212.

[17] E. Heine, Untersuchungen über die Reihe $1 + [(1-q^{\alpha})(1-q^{\beta})]/[(1-q)(1-q^{\gamma})]x + [(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})]/[(1-q)(1-q^{2})(1-q^{\gamma}(1-q^{\gamma+1})]x^{2} + \dots$ *J. reine angew. Math.* **34**, (1847) 285-328.

[18] F.H.Jackson, A basic-sine and cosine with symbolical solution of certain differential equations. *Proc. Edinburgh math. Soc.* **22** (*1904*), 28–39.

[19] F.H.Jackson, On $q$-functions and a certain difference operator. *Trans. Roy Soc. Edin.* **46** (*1908*), 253–281.

[20] F.H.Jackson, On basic double hypergeometric functions. Quart. J. Math., Oxford Ser. 13 (1942), 69–82.

[21] J. V. Kadeisvili, An introduction to the Lie-Santilli isotheory. *Algebras Groups Geom.* **13** (1996), no. 2, 171–281.

[22] P.Nalli, Sopra un procedimento di calcolo analogo alla integrazione. (Italian) Palermo Rend. 47, 337-374 (1923)

[23] Pascal,B, Traité du triangle arithmétique. Paris 1665.

[24] J.Riordan, Combinatorial identities. Reprint of the 1968 original. Robert E. Krieger Publishing Co., Huntington, N.Y., 1979.

[25] Rota, G.-C.; Taylor, B. D. The classical umbral calculus. *SIAM J. Math. Anal.* **25** (1994), no. 2, 694–711.

[26] R.M. Santilli, Imbedding of Lie algebras in nonassociative structures. *Nuovo Cimento,* X. Ser. A 51, 570-576 (1967).

[27] R.M. Santilli, Lie-admissible invariant treatment of irreversibility for matter and antimatter at the classical and operator levels. *Nuovo Cimento*, X. Ser. B 121, (2006).

[28] J.D.Talman, Special functions: A group theoretic approach. Based on lectures by Eugene P. Wigner. With an introduction by Eugene P. Wigner W. A. Benjamin, Inc., New York-Amsterdam 1968.

[29] M.Ward, A calculus of sequences. *Amer. J. Math.* **58** (1936), 255–266.

[30] J. Wess, B. Zumino, Covariant differential calculus on the quantum hyperplane. Nuclear Phys. B Proc. Suppl. 18B (1990), 302–312 (1991).

[31] A.Wawrzynczyk, Group representations and special functions. Examples and problems prepared by Aleksander Strasburger. Translated from the Polish by Bogdan Ziemian. Mathematics and its Applications (East European Series). D. Reidel Publishing Co., Dordrecht; PWN—Polish Scientific Publishers, Warsaw, 1984.

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