EXAMPLES OF q-ORTHOGONAL POLYNOMIALS

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1. Abstract

We first study four different types of q-Hermite polynomials

$$h_{\nu,q}(x), \psi_{\nu,q}(x), k_{\nu,q}(x), f_{\nu,q}(x)$$

from the point of view of generating functions and operational formulas. Three of these q-Hermite polynomials are q-Appell polynomials, and the remaining one is a pseudo q-Appell polynomial. q-difference equations and q-hypergeometric expressions are found. In this paper, we only obtain Rodriguez formulas for two of the four polynomials, namely $h_{\nu,q}(x)$ and $k_{\nu,q}(x)$, and accordingly we can only find q-analogues of the so-called Nielsen's formula for these two polynomials. Matrix forms for the polynomials expressed by q-Pascal matrices are considered. An orthogonality relation for $k_{\nu,q}(x)$ expressed as a q-integral is found. The two Cigler q-Laguerre polynomials are introduced and with the help of the relationship between $h_{\nu,q}(x)$ and the q-Laguerre polynomials, new generating functions for $h_{\nu,q}(x)$ are found. In the second part of the paper we study orthogonality relations for q-Jacobi-, q-Laguerre- and q-Legendre polynomials. The proofs will all use q-integration by parts, a method equivalent to the previously used recurrence technique. The orthogonality relations are all of discrete type.

2. Introduction

There are two kinds of Hermite polynomials, $H_n(x)$, and $He_n(x)$. The first one is defined by

$$e^{2xt-t^2} = \sum_{k=0}^{\infty} \frac{H_n(x)t^n}{n!}. \tag{1}$$

The second one is defined by

$$e^{xt-\frac{t^2}{2}} = \sum_{k=0}^{\infty} \frac{He_n(x)t^n}{n!}. \tag{2}$$

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The formulas for the two kinds of Hermite polynomials are quite similar; sometimes a formula only occurs in the literature in the shape of one of these polynomials. One example is Burchnall's formula [5, p. 9]. We will however try to make the best of it and treat these two kinds of Hermite polynomials as a unity. When we refer to an equation with an Hermite polynomial of the type (1), we usually denote this with a sentence like: almost a q-analogue of ..,

The Hermite polynomials were first studied by Sturm [32] in 1836, who proved that all its zeros are real. In 1864 Hermite [25] presented these polynomials in the form $H_n(x)$ with the Rodriguez formula, differential equation, and orthogonality.

The generating function (1) had been given already by Laplace in connection with the potential in his famous work about celestial mechanics. Halphen [24] studied the related Appell polynomials in his own way. This made the way for the three following crucial contributions. Laguerre [28] studied $He_n(x)$ and concluded that the Hermite polynomials give the successive derivatives of $e^{\frac{x^2}{2}}$ via the Rodriguez formula. In 1880 Appell [2, p. 122] gave a modern interpretation of Hermite polynomials in terms of Appell polynomials. W. Hahn [23] studied the polynomials in the form $He_n(x)$ and made a thorough study of its zeros.

The following formula and its inverse occur regularly in the literature. One of these was published in different form by Nielsen [30, p. 32].

Theorem 2.1. *Nielsen's formula* [5, p. 10], [19].

$$He_{n+r}(x) = \sum_{m=0}^{\min(r,n)} (-1)^m \binom{r}{m}\binom{n}{m} He_{r-m}(x) He_{n-m} m!. \qquad (3)$$

This formula has been very nicely proved by induction by Chatterjea [7, p. 53].

The four q-Hermite polynomials will first be defined by operator formulas. Then generating functions, recurrences, alternative operator formulas, q-difference equations and power series representations for the q-Hermite polynomials will be given. In this paper, we only obtain Rodriguez formulas for two of the four polynomials, namely $h_{\nu,q}(x)$ and $k_{\nu,q}(x)$. The Rodriguez formula is intimately connected with the orthogonality relation. Consequently only two orthogonality relations with q-integral representation are known so far for the four q-Hermite polynomials in this paper. The q-orthogonality for $h_{\nu,q}(x)$ was found in Kirschenhofer [27, p. 303]. In this paper we find the q-orthogonality for

$k_{\nu,q}(x)$ in (60). The generating functions for the four q-Hermite polynomials corresponding to (2) use a special q-exponential function with quadratic argument. An interesting lemma for computations with this q-exponential function is (15). We will also consider relations between q-Hermite polynomials and related polynomials like Carlitz-AlSalam and q-Laguerre.

Definition 1. The following symbols [11] will be used.

$$\{n\}_q!! \equiv \begin{cases} \prod_{k=1}^{\frac{n}{2}} \{2k\}_q, & \text{if } n \text{ even} \\ \prod_{k=1}^{\frac{n+1}{2}} \{2k-1\}_q, & \text{if } n \text{ odd} \end{cases} \tag{4}$$

$$\{n\}_{\alpha,q} \equiv \frac{\{n\alpha\}_q}{\{\alpha\}_q} \tag{5}$$

$$\{n\}_{\alpha,q}! \equiv \prod_{k=1}^{n} \frac{\{k\alpha\}_q}{\{\alpha\}_q} \tag{6}$$

$$\mathrm{E}_{\alpha,q}(x) \equiv \sum_{k=0}^{\infty} \frac{x^k}{\{k\}_{\alpha,q}!} \tag{7}$$

Theorem 2.2. *Cigler* [11, p. 30]. *The q-difference equation*

$$D_q f(x) = axf(x), \tag{8}$$

with boundary value $f(0) = 1$ has the solution

$$f(x) = \mathrm{E}_{2,q}\left(\tfrac{ax^2}{\{2\}_q}\right). \tag{9}$$

Remark 1. We often use q-exponential functions as weight functions in q-integrals. We then have

$$\lim_{x\to+\infty} \mathrm{E}_q(-x) = 0, \; 0 < |q| < 1. \tag{10}$$

On the the other hand, for $0 < |q| < 1$, the function

$\mathrm{E}_{\frac{1}{q}}(-x)$

oscillates around zero with decreasing amplitude for positive real values of x. The behaviour of the function $\mathrm{E}_{2,q}(\frac{x^2}{\{2\}_q})$ for $0 < |q| < 1$ as $\lim_{x\to\pm\infty}$ is very similar to that of $\mathrm{E}_q(-x)$ as $\lim_{x\to+\infty}$. For this reason these last two functions are used as weight functions for q-Hermite- and q-Laguerre polynomials.

Definition 2. The following q-analogue of [29] is a special kind of q-Appell polynomial. We will often use the generating function technique to define polynomials. The β_q polynomials of degree ν and order n are given by

$$\frac{t^n}{(\mathrm{E}_q(t)-1)^n} g(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \beta_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{11}$$

From (11), with $g(t) = \frac{1}{\mathrm{E}_{2,q}(\frac{qt^2}{\{2\}_q})}$, we conclude that $h_{\nu,q}(x)$ can be regarded as β_q polynomials of order 0,

$$\beta_{\nu,q}^{(0)}(x) \equiv h_{\nu,q}(x). \tag{12}$$

With this defintion, q-Hermite polynomials of order n are given by

$$\frac{t^n}{(\mathrm{E}_q(t)-1)^n} \frac{\mathrm{E}_q(xt)}{\mathrm{E}_{2,q}(\frac{qt^2}{\{2\}_q})} = \sum_{\nu=0}^{\infty} \frac{t^\nu h_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{13}$$

The corresponding q-Hermite numbers of order n are given by

$$\frac{t^n}{(\mathrm{E}_q(t)-1)^n} \frac{1}{\mathrm{E}_{2,q}(\frac{qt^2}{\{2\}_q})} = \sum_{\nu=0}^{\infty} \frac{t^\nu h_{\nu,q}^{(n)}}{\{\nu\}_q!}. \tag{14}$$

We are now going to define four q-Hermite polynomials. Two of them have been given before by Cigler [11], Désarménien [13] ($h_{n,q}(x)$), and Kirschenhofer [27] ($h_{n,q}(x), \psi_{n,q}(x)$). These four q-Hermite polynomials can be q-Appell polynomials, or pseudo q-Appell polynomials. One can see from the generating function which of these two classes the polynomial belongs to. The various formulas are going to be quite similar, and we are going to present the formulas in blocks to give a better overview. Matrix representations will also be given.

The following lemma will be used in the proof of Rodriguez formula for $k_{n,q}(x)$.

Lemma 2.3.

$$\mathrm{E}_{\frac{1}{q}}(xt)\mathrm{E}_{2,q}(\frac{-t^2}{\{2\}_q})\mathrm{E}_{2,q}(\frac{-x^2}{\{2\}_q}) = \mathrm{E}_{2,q}(\frac{-(x\ominus_q t)^2}{\{2\}_q}). \tag{15}$$

We start with

Definition 3. The Cigler q-Hermite polynomials [11] are defined by

$$h_{n,q}(x) \equiv (\mathbf{x} - q^{n-1}D_q)(\mathbf{x} - q^{n-2}D_q)\ldots(\mathbf{x} - qD_q)1. \tag{16}$$

The Kirschenhofer q-Hermite polynomials [27, p. 292] are given by

$$\psi_{n,q}(x) \equiv (\mathbf{x} - \epsilon D_q)^n 1. \tag{17}$$

The polynomial $k_{n,q}(x)$ is influenced by Cigler [11].

$$k_{n,q}(x) \equiv (\mathbf{x}\epsilon - D_q)^n 1. \tag{18}$$

$$f_{n,q}(x) \equiv (\mathbf{x} - q^2D_q)(\mathbf{x} - q^4D_q)\ldots(\mathbf{x} - q^{2n}D_q)1. \tag{19}$$

Theorem 2.4. *The first of the following generating functions is found in* [11, p. 42]*. Formula (22) gives an example of a pseudo q-Appell polynomial.*

$$\frac{\mathrm{E}_q(xt)}{\mathrm{E}_{2,q}(\frac{qt^2}{\{2\}_q})}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}h_{\nu,q}(x), \tag{20}$$

$$\mathrm{E}_q(xt)\mathrm{E}_{2,q}(\tfrac{-t^2}{\{2\}_q})=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\psi_{\nu,q}(x), \tag{21}$$

$$\mathrm{E}_{\frac{1}{q}}(xt)\mathrm{E}_{2,q}(\tfrac{-t^2}{\{2\}_q})=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}k_{\nu,q}(x). \tag{22}$$

$$\mathrm{E}_q(xt)\mathrm{E}_{2,q}(\tfrac{-t^2q^4}{\{2\}_q})q^{-2}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}f_{\nu,q}(x). \tag{23}$$

Theorem 2.5. *We have the following recurrences, all polynomials with index* 0 *are* 1*.*

$$[11, p.\ 43\ (4)]\ h_{\nu+1,q}(x)=xh_{\nu,q}(x)-q^\nu\{\nu\}_q h_{\nu-1,q}(x), \tag{24}$$

$$[27, p.\ 292]\ \psi_{\nu+1,q}(x)=x\psi_{\nu,q}(x)-\{\nu\}_q\psi_{\nu-1,q}(qx), \tag{25}$$

$$k_{\nu+1,q}(x)=xk_{\nu,q}(qx)-\{\nu\}_q k_{\nu-1,q}(qx), \tag{26}$$

$$f_{\nu+1,q}(x)=xf_{\nu,q}(x)-q^{\nu+1}\{\nu\}_q f_{\nu-1,q}(x). \tag{27}$$

Theorem 2.6. *The two first of the following four alternative operator representations are from Cigler* [11, p. 45]*.*

$$h_{\nu,q}(x)=q^{\binom{n}{2}}(\mathbf{x}\epsilon^{-1}-\epsilon^{-1}D_q)^n 1, \tag{28}$$

$$h_{\nu,q}(x)=(\mathbf{x}-D_q)(\mathbf{x}-q^3D_q)\ldots(\mathbf{x}-q^{2n-1}D_q)1, \tag{29}$$

$$\psi_{n,q}(x)=(\mathbf{x}-D_q)(\mathbf{x}-q^2D_q)\ldots(\mathbf{x}-q^{2n-2}D_q)1, \tag{30}$$

$$f_{n,q}(x)=(\mathbf{x}-q^nD_q)(\mathbf{x}-q^{n-1}D_q)\ldots(\mathbf{x}-qD_q)1. \tag{31}$$

Theorem 2.7. *The q-difference equations are*

$$(q^{\nu-1}D_q^2-xD_q+\{\nu\}_q)h_{\nu,q}(x)=0, \tag{32}$$

$$(q^{-2}D_q^2\epsilon-xD_q+\{\nu\}_q)\psi_{\nu,q}(x)=0, \tag{33}$$

$$(qD_q^2\epsilon^{-1}-xD_q+\{\nu\}_q)k_{\nu,q}(x)=0, \tag{34}$$

$$(q^\nu D_q^2-xD_q+\{\nu\}_q)f_{\nu,q}(x)=0. \tag{35}$$

Theorem 2.8. *A q-analogue of the Chatterjea operator formula* [8, p. 683, (3)].

$$h_{n,q}(x) = \frac{1}{(-x)^n}$$
$$(\mathbf{x}D_q - \mathbf{x}^2 - \{n-1\}_q)(\mathbf{x}D_q - \mathbf{x}^2 - \{n-2\}_q)\ldots(\mathbf{x}D_q - \mathbf{x}^2 - \{0\}_q)1. \tag{36}$$

Proof. Use induction. □

Theorem 2.9. *Explicit formulas for the four polynomials. We have written the first two in two different forms.*

$$\begin{gathered}[27, p.296]\ h_{n,q}(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}_q (-1)^k q^{k^2} \{2k-1\}_q!! x^{n-2k} \equiv \\ x^n {}_4\phi_1\left(-\tfrac{n}{2}, -\tfrac{n-1}{2}, \widetilde{-\tfrac{n}{2}}, \widetilde{-\tfrac{n-1}{2}}; \widetilde{1} | q, -\frac{q^{2n}}{x^2(1-q)}\right).\end{gathered} \tag{37}$$

$$\begin{gathered}[27, p.296]\ \psi_{n,q}(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}_q (-1)^k \{2k-1\}_q!! x^{n-2k} \equiv \\ x^n {}_6\phi_1\left(-\tfrac{n}{2}, -\tfrac{n-1}{2}, \widetilde{-\tfrac{n}{2}}, \widetilde{-\tfrac{n-1}{2}}, \infty, \infty; \widetilde{1} | q, -\frac{q^{2n+1}}{x^2(1-q)}\right).\end{gathered} \tag{38}$$

$$k_{n,q}(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}_q (-1)^k q^{\binom{n-2k}{2}} \{2k-1\}_q!! x^{n-2k}. \tag{39}$$

$$f_{n,q}(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}_q (-1)^k q^{2\binom{k+1}{2}} \{2k-1\}_q!! x^{n-2k}. \tag{40}$$

Proof. We prove the first formula.

$$\begin{aligned} LHS &= x^n \sum_i \frac{\langle 1; q\rangle_n \langle 1; q\rangle_{2i-1} (-1)^i x^{-2i} q^{i^2}}{\langle 1; q\rangle_{2i} \langle 1; q\rangle_{n-2i} \langle 1; q^2\rangle_{i-1} (1-q)^i} \\ &= x^n \sum_i \frac{\langle -n; q\rangle_{2i} q^{-\binom{2i}{2}+2in+i^2} (-1)^i x^{-2i}}{\langle 1; q^2\rangle_{i-1} (1-q)^i (1-q^{2i})} = \\ & x^n \sum_i \frac{\langle -\frac{n}{2}, -\frac{n-1}{2}, \widetilde{-\frac{n}{2}}, \widetilde{-\frac{n-1}{2}}; q\rangle_i (-1)^i x^{-2i} q^{-2\binom{i}{2}+2in}}{\langle 1, \widetilde{1}; q\rangle_i (1-q)^i}. \end{aligned} \tag{41}$$

□

Theorem 2.10. *A q-Rodriguez formula for $k_{\nu,q}(x)$.*

$$k_{\nu,q}(x) = (-1)^\nu (\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))^{-1} D_q^\nu \mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}). \tag{42}$$

Proof. A q-analogue of Appell, Kampé de Fériet [3, p. 334] and Rainville [31, p. 189]. This follows immediately from the first q-Taylor formula, (22) and (15). Another proof is just to use Cigler [11, p. 31, (8)]. □

Corollary 2.11. *Almost a q-analogue of Burchnall's operational formula* [5, p. 9]. *Let ϵ operate on the function $y(x)$. We assume that umbral calculus is used for the index of $k_{\nu,q}(x)$ in the NWA q-addition on the RHS. Then*

$$(\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))^{-1} D_q^n (\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q})y) = (-k_{.,q}(x)\epsilon \oplus_q D_q)^n y. \tag{43}$$

Proof. Use the q-Leibniz rule together with (42). □

In the same way follows

Theorem 2.12. *Cigler* [11, p. 45]. *The Rodriguez formula for $h_{\nu,q}(x)$, a q-analogue of* [3, p.334].

$$h_{\nu,q}(x) = (-1)^\nu q^{\binom{\nu}{2}} (\mathrm{E}_{2,\frac{1}{q}}(\tfrac{-qx^2}{\{2\}_q}))^{-1} D_{\frac{1}{q}}^\nu \mathrm{E}_{2,\frac{1}{q}}(\tfrac{-qx^2}{\{2\}_q}). \tag{44}$$

Theorem 2.13. *Another operational formula is Cigler* [11, p. 42, (3)], [13, p. 8, 6.3].

$$h_{\nu,q}(x) = \sum_{k=0}^{\infty} \frac{q^{k^2}(-D_q^2)^k}{\{2k\}_q!!} x^\nu. \tag{45}$$

Proof. Use (37). □

The following table lists the first five $h_{n,q}(x)$

1
x
$x^2 - q$
$x^3 - q\{3\}_q x$
$x^4 - q\binom{4}{2}_q x^2 + q^4\{3\}_q$
$x^5 - \binom{5}{2}_q x^3 q + q^4\{3\}_q\{5\}_q x$

The following table lists the first five $\psi_{n,q}(x)$

1
x
$x^2 - 1$
$x^3 - \{3\}_q x$
$x^4 - \binom{4}{2}_q x^2 + \{3\}_q$
$x^5 - \binom{5}{2}_q x^3 + \{3\}_q\{5\}_q x$

The following table lists the first five $k_{n,q}(x)$

1
x
x^2q-1
$x^3q^3-\{3\}_q x$
$x^4q^6-\binom{4}{2}_q x^2q+\{3\}_q$
$x^5q^{10}-\binom{5}{2}_q x^3q^3+\{3\}_q\{5\}_q x$

The following table lists the first five $f_{n,q}(x)$

1
x
x^2-q^2
$x^3-\{3\}_q xq^2$
$x^4-\binom{4}{2}_q q^2x^2+\{3\}_q q^6$
$x^5-\binom{5}{2}_q x^3q^2+\{3\}_q\{5\}_q xq^6$

As all these polynomials form (pseudo)q-Appell sequences it makes sense to define their vector forms. We will use the following vector forms for these polynomials

$$H_q(x) \equiv (h_{0,q}(x), h_{1,q}(x), \ldots, h_{n-1,q}(x))^T. \tag{46}$$

$$\Psi_q(x) \equiv (\psi_{0,q}(x), \psi_{1,q}(x), \ldots, \psi_{n-1,q}(x))^T. \tag{47}$$

$$K_q(x) \equiv (k_{0,q}(x), k_{1,q}(x), \ldots, k_{n-1,q}(x))^T. \tag{48}$$

$$F_q(x) \equiv (f_{0,q}(x), f_{1,q}(x), \ldots, f_{n-1,q}(x))^T. \tag{49}$$

Let the $n \times n$ q-matrix $H_{n,q}$ be given by

$$\begin{aligned} &H_{n,q}(i,i-1) \equiv \{i\}_q, \ i=1,\ldots,n-1, \\ &H_{n,q}(i,j) \equiv 0, \ j \neq i-1. \end{aligned} \tag{50}$$

Then

$$H_q(x) = P_{n,q}(x)H_q(0), \tag{51}$$

where the q-Pascal matrix $P_{n,q}(t) \equiv \mathrm{E}_q(H_{n,q}t)$ is given by the familiar expression [18]

$$P_{n,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!} H_{n,q}^k. \tag{52}$$

The same formula obtains for $\Psi_q(x)$ and $F_q(x)$. Finally

$$K_q(x) = \mathrm{E}_{\frac{1}{q}}(H_{n,q}t)K_q(0). \tag{53}$$

We will now come to formulas where one q-Hermite polynomial can be expressed as a weighted sum of a product of the same polynomials.

Theorem 2.14. *A q-analogue of (3).*

$$k_{n+r,q}(x) = \sum_{m=0}^{\min(r,n)} (-1)^m \binom{r}{m}_q \binom{n}{m}_q k_{r-m,q}(x) k_{n-m,q}(xq^r)\{m\}_q! q^{\binom{m}{2}}. \tag{54}$$

Proof. A q-analogue of Chatterjea [7]. We start with the following formula, modify it and find out that it is equal to the LHS.

$$\sum_{m=0}^{r} (-1)^m \binom{r}{m}_q k_{r-m,q}(x)\epsilon^{r-m} D_q^m k_{n,q}(x) = (-1)^r (\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))^{-1}$$
$$\sum_{m=0}^{r} \binom{r}{m}_q D_q^{r-m} \mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q})\epsilon^{r-m} D_q^m k_{n,q}(x) = (-1)^r (\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))^{-1}$$
$$D_q^r(\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))k_{n,q}(x) = (-1)^{n+r}(\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))^{-1} D_q^{n+r}\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}) = k_{n+r,q}(x). \tag{55}$$

On the other hand, the first expression can be simplied to

$$\sum_{m=0}^{\min(r,n)} (-1)^m \binom{r}{m}_q \binom{n}{m}_q k_{r-m,q}(x) k_{n-m,q}(xq^r)\{m\}_q! q^{\binom{m}{2}}. \tag{56}$$

□

Cigler [11, p. 45 (15)] has found a similar formula:

$$h_{n+r,q}(x) = q^{nr} \sum_{m=0}^{\min(r,n)} (-1)^m \binom{r}{m}_q \binom{n}{m}_q h_{r-m,q}(xq^{-n}) h_{n-m,q}(x)\{m\}_q!. \tag{57}$$

We now come to formulas which are some kind of inverses for the power series representations of q-Hermite polynomials.

Theorem 2.15. *Kirschenhofer* [27, p. 295]. *Almost a q-analogue of* [6, p. 370, (5.4)].

$$x^n = \sum_{2k \le n} q^k \{2k-1\}_q!! \binom{n}{2k}_q h_{n-2k,q}(x). \tag{58}$$

Proof. Use the generating function (20), multiply by $\mathrm{E}_{2,q}(\frac{qt^2}{\{2\}_q})$, and equate coefficients of t^n. □

Theorem 2.16. *A corrected version of Kirschenhofer* [27, p. 295]*. Almost a q-analogue of* [6, p. 370, (5.4)]*.*

$$x^n = \sum_{2k\leq n} q^{2\binom{k}{2}}\{2k-1\}_q!!\binom{n}{2k}_q \Psi_{n-2k,q}(x). \tag{59}$$

The following q-orthogonality has not been found before. The special form of q-Hermite polynomial chosen makes the proof quite simple.

Theorem 2.17. *q-orthogonality for $k_{n,q}(x)$.*

$$\int_{-\infty}^{\infty} k_{n,q}(x)k_{m,q}(x)\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q})\, d_q(x) = \delta(m,n)q^{-n}\{n\}_q!\int_{-\infty}^{\infty} \mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q})\, d_q(x). \tag{60}$$

Proof. Assume that $n \geq m$. q-Integration by parts gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} k_{n,q}(x)k_{m,q}(x)\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q})\, d_q(x) = \\
&\int_{-\infty}^{\infty} k_{m,q}(x)(-1)^n D_q^n(\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))\, d_q(x) = \\
&[\epsilon^{-1}k_{m,q}(x)(-1)^n D_q^{n-1}(\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))]_{-\infty}^{\infty} - \\
&\int_{-\infty}^{\infty} (D_q\epsilon^{-1})(k_{m,q}(x))(-1)^n D_q^{n-1}(\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q}))\, d_q(x) = \ldots = \\
&-\sum_{l=1}^{n} q^{-l+1}\{m-l+2\}_{l-1,q}[k_{m-l+1,q}(q^{-1}x)\mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q})k_{n-l,q}(x)]_{-\infty}^{\infty} + \\
&\delta(m,n)q^{-n}\{n\}_q!\int_{-\infty}^{\infty} \mathrm{E}_{2,q}(\tfrac{-x^2}{\{2\}_q})\, d_q(x).
\end{aligned} \tag{61}$$

□

The Carlitz-AlSalam orthogonal polynomials $F_{n,q}(x)$ [1], [13] are examples of q-analogues of x^n defined by a generating function made up of q-exponential functions.

Definition 4.

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}F_{\nu,q}(x) = \frac{\mathrm{E}_q(xt)}{\mathrm{E}_q(t)\mathrm{E}_q(-t)}. \tag{62}$$

These polynomials $F_{n,q}(x)$ are connected to the Cigler q-Hermite polynomials [11] by the following substitution. This substitution is only valid for $q \neq 1$.

$$h_{\nu,q}(x) = \left(\frac{q}{1-q}\right)^{\frac{\nu}{2}} F_{\nu,q}\left(x\sqrt{\frac{1-q}{q}}\right). \quad (63)$$

Now let's do the substitution (63) for the second q-Hermite polynomial. We start with $\psi_{\nu,q}(x)$ and obtain the following generating function for $\psi'_{\nu,q}(x)$, another q-analogue of x^n.

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\psi'_{\nu,q}(x) = \mathrm{E}_q(xt)\mathrm{E}_q(\tfrac{it}{\sqrt{q}} \ominus_q \tfrac{it}{\sqrt{q}}). \quad (64)$$

Here

$$\psi_{\nu,q}(x) = \left(\frac{q}{1-q}\right)^{\frac{\nu}{2}} \psi'_{\nu,q}\left(x\sqrt{\frac{1-q}{q}}\right). \quad (65)$$

The q-Laguerre polynomial $L^{(\alpha)}_{n,q,c}(x)$ was used by Cigler [11].

$$\begin{aligned} L^{(\alpha)}_{n,q,c}(x) &= (-1)^n(\epsilon - D_q)^{n+\alpha}x^n \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}_q \frac{\{n\}_q!}{\{k\}_q!} q^{k^2+\alpha k}(-1)^k x^k \\ &\equiv \sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{k^2+k}{2}+kn+\alpha k}(1-q)^k x^k}{(1-q)^n} \\ &\equiv \frac{\langle 1+\alpha;q\rangle_n}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}_q \frac{q^{k^2+\alpha k}(-x)^k(1-q)^k}{\langle 1+\alpha;q\rangle_k} \\ &\equiv \frac{\langle \alpha+1;q\rangle_n}{(1-q)^n}{}_1\phi_1\left(-n;\alpha+1|q,-x(1-q)q^{n+\alpha+1}\right). \end{aligned} \quad (66)$$

Cigler [10] also defined a closely related q-Laguerre polynomial $l^{(\alpha)}_{n,q,c}(x)$.

$$\begin{aligned} l^{(\alpha)}_{n,q,c}(x) &= (-1)^n(1 \boxminus_q D_q)^{n+\alpha}x^n \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}_q \frac{\{n\}_q!}{\{k\}_q!} q^{\binom{n-k}{2}}(-1)^k x^k \\ &\equiv \sum_{k=0}^{n} \frac{\langle 1+\alpha;q\rangle_n}{\langle 1+\alpha;q\rangle_k}\frac{\langle -n;q\rangle_k}{\langle 1;q\rangle_k}\frac{q^{\frac{n^2-n}{2}+k}(1-q)^k x^k}{(1-q)^n} \\ &\equiv \frac{\langle 1+\alpha;q\rangle_n}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}_q \frac{q^{\binom{n}{2}+k-nk+\binom{k}{2}}(-x)^k(1-q)^k}{\langle 1+\alpha;q\rangle_k} \\ &\equiv \frac{\langle \alpha+1;q\rangle_n}{(1-q)^n}q^{\binom{n}{2}}{}_2\phi_1\left(-n,\infty;\alpha+1|q,x(1-q)q\right). \end{aligned} \quad (67)$$

The following relation between the two polynomials obtains, as was pointed out by Richard Askey in his review [4].

$$l^{(\alpha)}_{n,q^{-1},c}(x) = q^{n-n^2-\alpha n} L^{(\alpha)}_{n,q,c}(xq^{-1}). \tag{68}$$

We can now express two of our q-Hermite polynomials in terms of small q-Laguerre polynomials. The first two of the following four equations are from Cigler [11, p. 55].

$$h_{2n,q}(x) = (-q(1+q))^n l^{(-\frac{1}{2})}_{n,q^2,c}(\frac{x^2q^{-1}}{1+q}), \tag{69}$$

$$h_{2n+1,q}(x) = x(-q(1+q))^n l^{(\frac{1}{2})}_{n,q^2,c}(\frac{x^2q^{-1}}{1+q}), \tag{70}$$

$$f_{2n,q}(x) = (-q^2(1+q))^n l^{(-\frac{1}{2})}_{n,q^2,c}(\frac{x^2q^{-2}}{1+q}), \tag{71}$$

$$f_{2n+1,q}(x) = x(-q^2(1+q))^n l^{(\frac{1}{2})}_{n,q^2,c}(\frac{x^2q^{-2}}{1+q}). \tag{72}$$

This implies

$$h_{2n,q}\left(\frac{\sqrt{x}(1+q)}{q}\right) = (-q(1+q))^n q^{-3n+2n^2} L^{(-\frac{1}{2})}_{n,q^{-2},c}(x), \tag{73}$$

$$h_{2n+1,q}\left(\frac{\sqrt{x}(1+q)}{q}\right) = (-q(1+q))^n q^{-n+2n^2} L^{(\frac{1}{2})}_{n,q^{-2},c}(x). \tag{74}$$

This gives the generating functions

$$\sum_{n=0}^{\infty} \frac{h_{2n,q}(\frac{\sqrt{x}(1+q)}{q})q^{3n-2n^2}t^n}{(-q(1+q))^n\{n\}_{q^{-2}}!} = \sum_{n=0}^{\infty} \frac{q^{-2n^2+n}(-xt)^n}{\{n\}_{q^{-2}}!(t;q^{-2})_{\frac{1}{2}+n}}, \tag{75}$$

$$\sum_{n=0}^{\infty} \frac{h_{2n+1,q}(\frac{\sqrt{x}(1+q)}{q})q^{n-2n^2}t^n}{(-q(1+q))^n\{n\}_{q^{-2}}!} = \sqrt{x}\sum_{n=0}^{\infty} \frac{q^{-2n^2-n}(-xt)^n}{\{n\}_{q^{-2}}!(t;q^{-2})_{\frac{3}{2}+n}}. \tag{76}$$

3. Orthogonality

In this chapter we consider orthogonality relations for q-Jacobi-, q-Laguerre- and q-Legendre polynomials. The proofs will all use q-integration by parts, a method equivalent to the previously used recurrence technique. The orthogonality relations are all of discrete type, a wellknown phenomenon.

We repeat the definition of q-Jacobi polynomials.

Definition 5.

$$P_{n,q}^{(\alpha,\beta)}(x) \equiv \frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n} {}_2\phi_1(-n,\beta+n;1+\alpha|q,xq^{\alpha+1-\beta}) \equiv \frac{\langle 1+\alpha;q\rangle_n}{\langle 1;q\rangle_n} \sum_{k=0}^{n} \binom{n}{k}_q \frac{\langle \beta+n;q\rangle_k}{\langle 1+\alpha;q\rangle_k} (-x)^k q^{\binom{k}{2}+(\alpha+1-\beta-n)k}. \tag{77}$$

Theorem 3.1. *The following Rodriguez formula obtains: Let* $x \in (0,|q^{\beta-\alpha-1}|)$. *Then*

$$P_{n,q}^{(\alpha,\beta)}(x) = \frac{x^{-\alpha}}{\{n\}_q!(xq^{\alpha+1-\beta};q)_{\beta-\alpha-1}} D_q^n \left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}} \right). \tag{78}$$

Theorem 3.2.

$$\int_0^{q^{\beta-\alpha-1}} P_{n,q}^{(\alpha,\beta)}(x) P_{m,q}^{(\alpha,\beta)}(x) x^{\alpha} \{n\}_q!(xq^{-\beta+\alpha+1};q)_{\beta-\alpha-1}\, d_q(x) = \\ = \delta(m,n) \frac{\langle \beta+n;q\rangle_n}{(1-q)^n} \mathrm{QE}((1+\alpha)(-\alpha+\beta+n)) B_q(\beta-\alpha+n,\alpha+1+n). \tag{79}$$

Proof. q-integration by parts gives

$$\int_0^{q^{\beta-\alpha-1}} P_{n,q}^{(\alpha,\beta)}(x) P_{m,q}^{(\alpha,\beta)}(x) x^{\alpha} \{n\}_q!(xq^{-\beta+\alpha+1};q)_{\beta-\alpha-1}\, d_q(x) =$$

$$\int_0^{q^{\beta-\alpha-1}} D_q^n\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right) P_{m,q}^{(\alpha,\beta)}(x)\, d_q(x) =$$

$$\left[D_q^{n-1}\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right)\epsilon^{-1} P_{m,q}^{(\alpha,\beta)}(x)\right]_0^{q^{\beta-\alpha-1}} -$$

$$\int_0^{q^{\beta-\alpha-1}} D_q^{n-1}\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right) D_q \epsilon^{-1} P_{m,q}^{(\alpha,\beta)}(x)\, d_q(x) = \ldots =$$

$$\sum_{l=1}^{n} (-1)^{l+1} \left[D_q^{n-l}\left(\frac{x^{\alpha+n}}{(x;q)_{\alpha+1-\beta-n}}\right)(\epsilon^{-1}D_q)^{l-1}\epsilon^{-1} P_{m,q}^{(\alpha,\beta)}(x)\right]_0^{q^{\beta-\alpha-1}} +$$

$$(-1)^n \int_0^{q^{\beta-\alpha-1}} x^{\alpha+n}(xq^{-\beta+\alpha+1-n};q)_{\beta+n-\alpha-1}(D_q\epsilon^{-1})^n [P_{m,q}^{(\alpha,\beta)}(x)]\, d_q(x) =$$

$$\sum_{l=1}^{n} [(-1)^{l+1} \sum_{k=0}^{n-l} (\binom{n-l}{k}_q \frac{\{-\beta-n+\alpha+1\}_{k,q}\{\alpha+1+k+l\}_{n-l-k,q} x^{\alpha+k+l}}{(xq^{n-k-l};q)_{\alpha+1-\beta+k-n}}) \times$$

$$(\epsilon^{-1}D_q)^{l-1}\epsilon^{-1} P_{m,q}^{(\alpha,\beta)}(x)]_0^{q^{\beta-\alpha-1}} +$$

$$(-1)^n \int_0^{q^{\beta-\alpha-1}} x^{\alpha+n}(xq^{-\beta-n+\alpha+1};q)_{\beta-\alpha-1}(D_q\epsilon^{-1})^n[P_{m,q}^{(\alpha,\beta)}(x)]\,d_q(x) = RHS.$$

The q-integral can be computed as follows.

$$\begin{aligned}
&\int_0^{q^{\beta-\alpha-1}} x^{\alpha+n}(xq^{-\beta+\alpha+1-n};q)_{\beta-\alpha-1+n}\,d_q(x) = \\
&q^{\beta-\alpha-1}(1-q)\sum_{m=n+1}^{\infty}\langle m-n;q\rangle_{n-\alpha+\beta-1}q^{(n+\alpha)(-\alpha+\beta+m-1)+m} = \\
&q^{\beta-\alpha-1}(1-q)\sum_{m=n+1}^{\infty}\frac{\langle m-n;q\rangle_\infty}{\langle m-\alpha+\beta-1;q\rangle_\infty}q^{(n+\alpha)(-\alpha+\beta+m-1)+m} = \\
&q^{\beta-\alpha-1}(1-q)\sum_{l=0}^{\infty}\frac{\langle l+1;q\rangle_\infty}{\langle l+n-\alpha+\beta;q\rangle_\infty}q^{(n+\alpha)(-\alpha+\beta+l+n)+l+1+n} = \\
&q^{\beta-\alpha-1}(1-q)\sum_{l=0}^{\infty}\frac{\langle 1;q\rangle_\infty\langle n-\alpha+\beta;q\rangle_l}{\langle n-\alpha+\beta;q\rangle_\infty\langle 1;q\rangle_l}q^{(n+\alpha)(-\alpha+\beta+l+n)+l+1+n} = \\
&q^{(\beta-\alpha+n)(n+\alpha+1)}(1-q)\frac{\langle 1,2n+\beta+1;q\rangle_\infty}{\langle n-\alpha+\beta,n+\alpha+1;q\rangle_\infty} = \\
&B_q(\beta-\alpha+n,\alpha+1+n)q^{(\beta-\alpha+n)(n+\alpha+1)}.
\end{aligned} \tag{80}$$

□

The orthogonality for q-Laguerre polynomials has a weight function which consists of x^α times a q-exponential function with negative function argument. This q-exponential function can also be written as an inverse q-shifted factorial $\frac{1}{(x(1-q);q)_\infty}$

We can use the definition of $E_q(-x)$ for $x < \frac{1}{1-q}$. For larger x we use the inverse q-shifted factorial formula.

Theorem 3.3. *An inequality for $E_q(-x)$.*

$$E_q(-x) > e^{-x}, 0 < q < 1, \quad x > 0. \tag{81}$$

Proof. Denote

$$P_N \equiv \prod_{k=0}^{N}\frac{1}{1+x(1-q)q^k}. \tag{82}$$

Then

$$P_N > exp(-\sum_{k=0}^{N}x(1-q)q^k) = exp(-x(1-q^N)) \tag{83}$$

Now

$$E_q(-x) = \lim_{N\to\infty} P_N > e^{-x}. \tag{84}$$

□

To do the complete proof of the following theorem, we need a formula for a certain q-integral.

Lemma 3.4. *Compare Jackson* [26, p. 200, (22)]. *The moments of order n for the q-Laguerre weight function are given by*

$$\int_0^\infty x^{\alpha+n} E_q(-x)\, d_q(x) = \mathrm{QE}\left(-\binom{n+\alpha+1}{2}\right)\Gamma_q(n+\alpha+1). \tag{85}$$

Since the Stieltje moment problem for q-Laguerre polynomials is indeterminate, there are many orthogonality relations. One of these is the following.

Theorem 3.5. *A q-analogue of* [33, p. 214, (1.6)]. *Let* Re $\alpha > -1$. *Then*

$$\begin{aligned}\int_0^\infty L_{n,q}^{(\alpha)}(x) L_{m,q}^{(\alpha)}(x) x^\alpha E_q(-x)\, d_q(x) = \delta(m,n)\frac{1}{\{n\}_q!}\times \\ \mathrm{QE}\left(\binom{n}{2} + n\alpha - \binom{n+\alpha+1}{2}\right)\Gamma_q(n+\alpha+1).\end{aligned} \tag{86}$$

Proof. Assume that $n \geq m$. q-Integration by parts gives

$$\int_0^\infty L_{n,q}^{(\alpha)}(x) L_{m,q}^{(\alpha)}(x) x^\alpha \{n\}_q! E_q(-x)\, d_q(x) =$$

$$\int_0^\infty L_{m,q}^{(\alpha)}(x) D_q^n(x^{\alpha+n} E_q(-x))\, d_q(x) =$$

$$[\epsilon^{-1} L_{m,q}^{(\alpha)}(x) D_q^{n-1}(x^{\alpha+n} E_q(-x))]_0^\infty -$$

$$\int_0^\infty (D_q\epsilon^{-1})(L_{m,q}^{(\alpha}(x)) D_q^{n-1}(x^{\alpha+n} E_q(-x))\, d_q(x) = \ldots =$$

$$\sum_{l=1}^n (-1)^{l+1}[(\epsilon^{-1}D_q)^{l-1}(\epsilon^{-1} L_{m,q}^{(\alpha)}(x)) D_q^{n-l}(x^{\alpha+n} E_q(-x))]_0^\infty +$$

$$(-1)^n \int_0^\infty (D_q\epsilon^{-1})^n [L_{m,q}^{(\alpha)}(x)] x^{\alpha+n} E_q(-x)\, d_q(x) =$$

$$\sum_{l=1}^n [(-1)^{l+1} \sum_{k=0}^{n-l} (\binom{n-l}{k}_q \{\alpha+1+k+l\}_{n-l-k,q} x^{\alpha+k+l}(-1)^k \times$$

$$q^{k(\alpha+k+l)}E_q(-x)(\epsilon^{-1}D_q)^{l-1}\epsilon^{-1}L_{m,q}^{(\alpha)}(x)]_0^\infty+$$

$$(-1)^n\int_0^\infty (D_q\epsilon^{-1})^n[L_{m,q}^{(\alpha)}(x)]x^{\alpha+n}E_q(-x)\,d_q(x)=$$

$$\delta(m,n)\mathrm{QE}(\binom{n}{2}+n\alpha)\int_0^\infty x^{\alpha+n}E_q(-x)\,d_q(x).$$

Finally use the lemma to complete the proof. □

Lemma 3.6.

$$D_q^k(1\boxplus_q x)^l=\left(\prod_{j=0}^{k-1}\{l-j\}_q\right)\ q^{\binom{k}{2}}(1\boxplus_q q^kx)^{l-k},\ \ l\geq k. \tag{87}$$

The following polynomial is defined by the Rodrigues formula to enable an easy orthogonality relation. q-Legendre polynomials have been given before, but these don't have the same orthogonality range in the limit $q\to 1$ as in the classical case. To be able to treat orthogonality properly, we only consider the Rodriguez formula.

Definition 6. The q-Legendre polynomial is defined by

$$P_{n,q}(x)\equiv\frac{q^{-\binom{n}{2}}(-1)^n}{\{n\}_q!(1\boxplus_q q^{-n})^n}D_q^n\left((1\boxminus_q x)^n(1\boxplus_q x)^n\right). \tag{88}$$

This implies

Theorem 3.7. *An explicit combinatorial formula for q-Legendre polynomials:*

$$\begin{aligned}P_{n,q}(x)=\frac{q^{-\binom{n}{2}}(-1)^n}{\{n\}_q!(1\boxplus_q q^{-n})^n}\sum_{k=0}^n\binom{n}{k}_q\prod_{m=0}^{k-1}(\{n-m\}_q)\ q^{\binom{k}{2}}(-1)^k\\(1\boxminus_q q^nx)^{n-k}q^{\binom{n-k}{2}}(1\boxplus_q q^{n-k}x)^k\prod_{l=0}^{n-k-1}\{n-l\}_q.\end{aligned} \tag{89}$$

Theorem 3.8. *For simplicity we put*

$$\widetilde{P_{n,q}(x)}\equiv D_q^n\left((1\boxminus_q x)^n(1\boxplus_q x)^n\right). \tag{90}$$

Orthogonality relation for q-Legendre polynomials:

$$\int_{-q^{1-m}}^{q^{1-m}} \widetilde{P_{m,q}(x)}\widetilde{P_{n,q}(x)}\, d_q(x) = \delta(m,n)(-1)^n \int_{-q^{1-m}}^{q^{1-m}} \sum_{k=0}^{m-n} \binom{m-n}{k}_q$$
$$\prod_{l=0}^{k-1}\{m-l\}_q\ q^{\binom{k}{2}}(-1)^k(1 \boxminus_q q^{m-n}x)^{m-k} q^{\binom{m-n-k}{2}} \frac{q^{-\binom{m}{2}}(-1)^m}{\{m\}_q!(1 \boxplus_q q^{-m})^m}$$
$$(1 \boxplus_q q^{m-n-k}x)^{k+n} \prod_{l=0}^{m-n-k-1} \{m-l\}_q D_q^n \widetilde{P_{n,q}(xq^{-n})}\, d_q(x). \tag{91}$$

Proof. q-integration by parts gives

$$\int_{-q^{1-m}}^{q^{1-m}} \widetilde{P_{m,q}(x)}\widetilde{P_{n,q}(x)}\, d_q(x) =$$
$$\int_{-q^{1-m}}^{q^{1-m}} D_q^m\left((1 \boxminus_q x)^m (1 \boxplus_q x)^m\right) \widetilde{P_{n,q}(x)}\, d_q(x) =$$
$$[D_q^{m-1}\left((1 \boxminus_q x)^m (1 \boxplus_q x)^m\right) \widetilde{P_{n,q}(xq^{-1})}]_{-q^{1-m}}^{q^{1-m}} -$$
$$\int_{-q^{1-m}}^{q^{1-m}} D_q^{m-1}\left((1 \boxminus_q x)^m (1 \boxplus_q x)^m\right) D_q \widetilde{P_{n,q}(xq^{-1})}\, d_q(x) = \ldots =$$
$$\sum_{k=1}^{n}(-1)^{k+1}[\sum_{l=0}^{m-k}\binom{m-k}{l}_q \prod_{j=0}^{l-1}\{m-j\}_q\, q^{\binom{l}{2}}(-1)^l(1\boxminus_q q^{m-k}x)^{m-l} q^{\binom{m-l-k}{2}}$$
$$(1 \boxplus_q q^{m-l-k}x)^{k+l} \prod_{j=0}^{m-l-k-1}\{m-j\}_q D_q^{k-1} \widetilde{P_{n,q}(q^{-k}x)}]_{-q^{1-m}}^{q^{1-m}} +$$
$$+\delta(m,n)(-1)^n$$
$$\int_{-q^{1-m}}^{q^{1-m}} \sum_{k=0}^{m-n}\binom{m-n}{k}_q \prod_{l=0}^{k-1}\{m-l\}_q\ q^{\binom{k}{2}}(-1)^k(1 \boxminus_q q^{m-n}x)^{m-k}$$
$$q^{\binom{m-n-k}{2}}(1 \boxplus_q q^{m-n-k}x)^{k+n} \prod_{l=0}^{m-n-k-1}\{m-l\}_q D_q^n \widetilde{P_{n,q}(xq^{-n})}\, d_q(x).$$

□

Theorem 3.9. *The q-Legendre polynomials $P_{n,q}(x)$ for small index are solutions of the following q-difference equations:*

$$(1-x^2)D_q^2 f(x,q) - \{2\}_q x D_q f(x,q) + \{2\}_q f(x,q) = 0 \tag{92}$$

has solution $f(x,q) = P_{1,q}(x)$.

$$(1-x^2q^2)D_q^2 f(x,q) - q^3\{2\}_q x D_q f(x,q) + q^2\{3\}_q! f(x,q) = 0 \quad (93)$$

has solution $f(x,q) = P_{2,q}(x)$.

$$(1-x^2q^6)D_q^2 f(x,q) - q^3\{2\}_q x D_q f(x,q) + q^3\{3\}_q(\{2\}_q)^2(q^2-q+1) f(x,q) = 0 \quad (94)$$

has solution $f(x,q) = P_{3,q}(x)$.

Theorem 3.10. *The function* $P_{n,q}(x)$ *is the solution of the following linear second order* q*-difference equation with initial value* $f(q^{-n}) = 1$:

$$(x^2q^{2n+2}-1)D_q^2 f(x,q) + q^n(\{2\}_q\{n+1\}_q - q\{2n\}_q) x D_q f(x,q) - q^n\{n\}_q\{n+1\}_q f(x,q) = 0. \quad (95)$$

Proof. A q-analogue of [12, p. 73]. Let

$$u = (-1)^n (x^2; q^2)_n \quad (96)$$

Then

$$(x^2-1)D_q u = \{2n\}_q x u. \quad (97)$$

Operate with D_q^{n+1} on this, and use the q-Leibniz theorem to obtain (95). ∎

4. Conclusion

We have brought together the two Cigler q-Hermite polynomials, the Kirschenhofer q-Hermite polynomial and a new q-Hermite polynomial to form a sequence of equations, where most of the formulas appear in quadruple. The near q-analogue (43) of Burchnall's operator formula [5, p. 9] is presented here for the first time. The q-exponential function $\mathrm{E}_{2,q}(\frac{-x^2}{\{2\}_q})$ introduced already by Jackson in his articles about q-Bessel functions around 1904, and also favoured by Cigler, plays a crucial role in this formula. The Rodriguez formula, imperative for (43) also involves the function $\mathrm{E}_{2,q}(\frac{-x^2}{\{2\}_q})$. The weight function for the q-orthogonality of $k_{n,q}(x)$ yet again involves $\mathrm{E}_{2,q}(\frac{-x^2}{\{2\}_q})$. We have found some operational formulas here, hopefully generalizations to other forms will be found in future papers. The work on q-orthogonalites will be continued in future papers.

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