q-TAYLOR FORMULAS WITH q-INTEGRAL REMAINDER

THOMAS ERNST

Abstract. We present four q-Taylor formulas with q-integral remainder. All proofs use q-integration by parts. The two last proofs require a slight rearrangement by a well-known formula. The two first formulas have been given in different form by Annaby and Mansour [2].

In a recent paper [2] Annaby and Mansour have found two q-Taylor formulas with q-integral remainder: (4) and a formula similar to (13). We are going to give a much simpler proof of (4), together with (13) and two other q-Taylor formulas which use the two types of q-addition. All proofs use q-integration by parts. To be able to work freely, we consider functions in $\mathbb{C}[[x]]$.

Lemma 0.1.

$$D_{q,t}\left(-\frac{P_{m,q}(x,-t)}{\{m\}_q!}\right)=\frac{P_{m-1,q}(x,-qt)}{\{m-1\}_q!},\ m=1,2,3,\ldots. \tag{1}$$

$$D_{q,t}^k(x\boxplus_q t)^l=\{l-k+1\}_{k,q}q^{\binom{k}{2}}(x\boxplus_q q^k t)^{l-k},\ l\geq k. \tag{2}$$

Theorem 0.2. *The first Jackson q-Taylor formula* [8, p. 63].

$$F(x)=\sum_{n=0}^{\infty}\frac{(x\boxminus_q y)^n}{\{n\}_q!}D_q^n F(y). \tag{3}$$

We will give a formal power series treatment in this paper and not use arguments involving type and order.

Compare [2, p. 476 3.5], where an equivalent form of the next theorem was given. The proof in [2] is however quite different.

Theorem 0.3. *Let $0<|q|<1$ and let f be n times q-differentiable in the closed interval $[a,x]$. Then the following generalization of Jackson's*

Date: June 1, 2009.
[0]2000 Mathematics Subject Classification: Primary 41A58; Secondary 33D15

formula holds for $n = 1, 2, \ldots$

$$f(x) = \sum_{k=0}^{n-1} \frac{P_{k,q}(x, -a)}{\{k\}_q!}(D_q^k f)(a) + \int_{t=a}^{x} \frac{P_{n-1,q}(x, -qt)}{\{n-1\}_q!}(D_q^n f)(t)\, d_q(t). \tag{4}$$

First proof: See my thesis [5]. Second proof:

Proof. The proof is very straightforward, at each step we only use q-integration by parts. We start with

$$f(x) = f(a) + \int_{t=a}^{x} D_q f(t) D_{q,t}(t-x)\, d_q(t), \tag{5}$$

which follows from the definition of q-integral. At the next step we obtain

$$f(x) = f(a) + [D_q f(t)(t-x)]_{t=a}^{t=x} - \int_{t=a}^{x} (qt - x) D_q^2 f(t)\, d_q(t). \tag{6}$$

We proceed with

$$f(x) = f(a) + D_q f(a)(x-a) - \left[D_q^2 f(t) \frac{1}{\{2\}_q!}(qt^2 - xt(1+q) + x^2) \right]_{t=a}^{t=x} + \int_{t=a}^{x} D_q^3 f(t) \frac{P_{2,q}(x, -qt)}{\{2\}_q!}\, d_q(t). \tag{7}$$

In the third step we obtain

$$\begin{aligned} f(x) = f(a) + D_q f(a) P_{1,q}(x, -a) + D_q^2 f(a) \frac{1}{\{2\}_q!} P_{2,q}(x, -a) - \\ \left[D_q^3 f(t) \frac{P_{3,q}(x, -t)}{\{3\}_q!} \right]_{t=a}^{t=x} + \int_{t=a}^{x} D_q^4 f(t) \frac{P_{3,q}(x, -qt)}{\{3\}_q!}\, d_q(t). \end{aligned} \tag{8}$$

We can continue this process forever. □

Theorem 0.4. *The q-Maclaurin theorem. Let* $0 < |q| < 1$ *and let* f *be* n *times q-differentiable in the closed interval* $[0, x]$*. Then the following formula holds for* $n = 1, 2, \ldots$

$$f(x) = \sum_{k=0}^{n-1} \frac{x^k}{\{k\}_q!}(D_q^k f)(0) + \int_{t=0}^{x} \frac{P_{n-1,q}(x, -qt)}{\{n-1\}_q!}(D_q^n f)(t)\, d_q(t). \tag{9}$$

Proof. Put $a = 0$ in (4). □

Example 1. We define a function with the property $D_q f(x) = (1 \boxplus_q x)^{-1}$ and $f(0) = 0$. We infer that

$$D_q^k f(x) = (-1)^{k-1} q^{1-k} \{k-1\}_q! (1 \boxplus_q xq^{k-1})^{-k}, \tag{10}$$

$$D_q^k f(0) = (-1)^{k-1} q^{1-k} \{k-1\}_q!. \tag{11}$$

We infer that

$$\begin{aligned} f(x) = \sum_{k=1}^{n-1} \frac{x^k}{\{k\}_q} (-1)^{k-1} q^{1-k} + \\ \int_{t=0}^{x} P_{n-1,q}(x, -qt)(-1)^{n-1} q^{1-n} (1 \boxplus_q tq^{n-1})^{-n} \, d_q(t). \end{aligned} \tag{12}$$

This is one q-analogue of $\ln(x+1)$.

A different form of the following theorem occurred in [2, p. 480 4.6].

Theorem 0.5. *Let $0 < |q| < 1$ and let f be n times q-differentiable in the closed interval $[a, x]$. Then the following generalization of Jackson's formula holds for $n = 1, 2, \ldots$:*

$$\begin{aligned} f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k q^{-\binom{k}{2}} P_{k,q}(a, -x)}{\{k\}_q!} (D_q^k f)(aq^{-k}) + \\ \int_{t=a}^{x} \frac{(-1)^{n-1} q^{-\binom{n}{2}} P_{n-1,q} t, -x)}{\{n-1\}_q!} (D_q^n f)(tq^{-n}) \, d_q(t). \end{aligned} \tag{13}$$

Proof. We use q-integration by parts We start with

$$f(x) = f(a) + \int_{t=a}^{x} D_q f(t) D_{q,t}(t-x) \, d_q(t), \tag{14}$$

which follows from the definition of q-integral. At the next step we obtain

$$f(x) = f(a) + \left[D_q f(tq^{-1})(t-x) \right]_{t=a}^{t=x} - \int_{t=a}^{x} q^{-1}(t-x) D_q^2 f(tq^{-1}) \, d_q(t). \tag{15}$$

We proceed with

$$\begin{aligned} f(x) = f(a) + D_q f(aq^{-1})(x-a) - \left[D_q^2 f(tq^{-2}) \frac{q^{-1}}{\{2\}_q!} (t^2 - xt(1+q) + qx^2) \right]_{t=a}^{t=x} \\ + \int_{t=a}^{x} D_q^3 f(tq^{-2}) \frac{q^{-3} P_{2,q}(t, -x)}{\{2\}_q!} \, d_q(t). \end{aligned} \tag{16}$$

In the third step we obtain

$$f(x)=f(a)-D_qf(aq^{-1})P_{1,q}(a,-x)+D_q^2f(aq^{-2})\frac{1}{\{2\}_q!}P_{2,q}(a,-x)+\\ \left[D_q^3f(tq^{-3})\frac{q^{-3}P_{3,q}(t,-x)}{\{3\}_q!}\right]_{t=a}^{t=x}-\int_{t=a}^{x}D_q^4f(tq^{-4})\frac{q^{-6}P_{3,q}(t,-x)}{\{3\}_q!}\,d_q(t). \tag{17}$$

We can continue this process forever. ☐

We are going to use the following formula in the last proofs:

Theorem 0.6. *Rothe 1811* [11]*, von Grüson* [13, S. 36] *1814, Gauß 1876* [6]*.*

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}u^n=(u;q)_m. \tag{18}$$

Our next aim is to find q-Taylor expansions with q-integral remainder for formulas corresponding to Nalli–Ward [10, p. 345], [15, p. 259] and Jackson [9, (51), p.77], respectively. These formulas are (29) and (33). We first prove preliminary lemmata (19) and (24), and then show by (18) that these are equivalent to the formulas we want to prove.

Lemma 0.7.

$$F(x\oplus_q y)=F(x)+\sum_{k=1}^{n-1}\frac{y^k}{\{k\}_q!}(-1)^{k+1}q^{\binom{k}{2}}D_{q,y}^kF(x\oplus_q y)+\\ \int_{t=0}^{y}D_{q,t}^n\left[F(x\oplus_q t)\right]\frac{(-t)^{n-1}}{\{n-1\}_q!}q^{\binom{n}{2}}\,d_q(t). \tag{19}$$

Proof. The proof is very straightforward, at each step we only use integration by parts. We start with

$$(x\oplus_q y)^m=x^m+\int_{t=0}^{y}D_{q,t}\left[(x\oplus_q t)^m\right]D_{q,t}(t)\,d_q(t), \tag{20}$$

which follows from the definition of q-integral. At the next step we obtain

$$(x\oplus_q y)^m=x^m+\left[D_{q,t}\left[(x\oplus_q t)^m\right]t\right]_{t=0}^{t=y}-\int_{t=0}^{y}qtD_{q,t}^2\left[(x\oplus_q t)^m\right]\,d_q(t). \tag{21}$$

We proceed with

$$(x \oplus_q y)^m = x^m + yD_{q,y}((x \oplus_q y)^m) - \left[D^2_{q,t}\left[(x \oplus_q t)^m\right]\frac{qt^2}{\{2\}_q!}\right]_{t=0}^{t=y}$$
$$+ \int_{t=0}^{y} D^3_{q,t}\left[(x \oplus_q t)^m\right]\frac{q^3t^2}{\{2\}_q!}\,d_q(t). \tag{22}$$

In the third step we obtain

$$(x \oplus_q y)^m = x^m + yD_{q,y}(x \oplus_q y)^m - D^2_{q,y}\left[(x \oplus_q y)^m\right]\frac{qy^2}{\{2\}_q!} +$$
$$\left[D^3_{q,t}\left[(x \oplus_q t)^m\right]\frac{q^3t^3}{\{3\}_q!}\right]_{t=0}^{t=y} - \int_{t=0}^{y} D^4_{q,t}\left[(x \oplus_q t)^m\right]\frac{q^6t^3}{\{3\}_q!}\,d_q(t). \tag{23}$$

We can continue this process forever. □

Lemma 0.8.

$$F(x \boxplus_q y) = F(x) + \sum_{k=1}^{n-1}\frac{y^k}{\{k\}_q!}(-1)^{k+1}q^{\binom{k}{2}}D^k_{q,y}F(x \boxplus_q y) +$$
$$\int_{t=0}^{y} D^n_{q,t}\left[F(x \boxplus_q t)\right]\frac{(-t)^{n-1}}{\{n-1\}_q!}q^{\binom{n}{2}}\,d_q(t). \tag{24}$$

Proof. The proof is almost the same as the previous one. We start with

$$(x \boxplus_q y)^m = x^m + \int_{t=0}^{y} D_{q,t}\left[(x \boxplus_q t)^m\right]D_{q,t}(t)\,d_q(t), \tag{25}$$

which follows from the definition of q-integral. At the next step we obtain

$$(x \boxplus_q y)^m = x^m + \left[D_{q,t}\left[(x \boxplus_q t)^m\right]t\right]_{t=0}^{t=y} - \int_{t=0}^{y} qtD^2_{q,t}\left[(x \boxplus_q t)^m\right]\,d_q(t). \tag{26}$$

We proceed with

$$(x \boxplus_q y)^m = x^m + yD_{q,y}((x \boxplus_q y)^m) - \left[D^2_{q,t}\left[(x \boxplus_q t)^m\right]\frac{qt^2}{\{2\}_q!}\right]_{t=0}^{t=y}$$
$$+ \int_{t=0}^{y} D^3_{q,t}\left[(x \boxplus_q t)^m\right]\frac{q^3t^2}{\{2\}_q!}\,d_q(t). \tag{27}$$

In the third step we obtain

$$
\begin{aligned}
&(x \boxplus_q y)^m = x^m + yD_{q,y}(x \boxplus_q y)^m - D^2_{q,y}\left[(x \boxplus_q y)^m\right]\frac{qy^2}{\{2\}_q!} + \\
&\left[D^3_{q,t}\left[(x \boxplus_q t)^m\right]\frac{q^3t^3}{\{3\}_q!}\right]_{t=0}^{t=y} - \int_{t=0}^{y} D^4_{q,t}\left[(x \boxplus_q t)^m\right]\frac{q^6t^3}{\{3\}_q!}\,d_q(t).
\end{aligned} \tag{28}
$$

We can continue this process forever. □

Theorem 0.9. *Compare with the Nalli–Ward q-Taylor formula.* [10, p. 345], [15, p. 259], *which is obtained by letting* $n \to \infty$.

$$
\begin{aligned}
&F(x \oplus_q y) = \sum_{k=0}^{n-1} \frac{y^k}{\{k\}_q!} D^k_q F(x) + \\
&\int_{t=0}^{y} D^n_{q,t}\left[F(x \oplus_q t)\right]\frac{(-t)^{n-1}}{\{n-1\}_q!} q^{\binom{n}{2}}\,d_q(t).
\end{aligned} \tag{29}
$$

Proof. We show that this is equivalent with (19). By putting $F(x) = x^m$ it would suffice to prove that

$$
\sum_{n=1}^{m} \frac{y^n}{\{n\}_q!} D^n_q x^m = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!}(-1)^{k+1} q^{\binom{k}{2}} D^k_{q,y}(x \oplus_q y)^m. \tag{30}
$$

This is equivalent to the formula

$$
\begin{aligned}
&\sum_{n=1}^{m} \frac{y^n}{\{n\}_q!}\{m-n+1\}_{n,q} x^{m-n} = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!}(-1)^{k+1} q^{\binom{k}{2}}\{m-k+1\}_{k,q} \\
&\sum_{l=0}^{m-k} \frac{\{m-k\}_q!}{\{m-k-l\}_q!\{l\}_q!} x^l y^{m-k-l}.
\end{aligned} \tag{31}
$$

By equating the corresponding exponents for x and y, and thus putting $n = m - l$ we obtain

$$
\frac{\{l+1\}_{m-l,q}}{\{n\}_q!} = \sum_{k=1}^{m-l} \frac{(-1)^{k+1} q^{\binom{k}{2}}\{m-k+1\}_{k,q}}{\{k\}_q!} \frac{\{m-k\}_q!}{\{m-k-l\}_q!\{l\}_q!}. \tag{32}
$$

After simplification we see that this is equivalent to (18) for the special case $u = 1$. □

Theorem 0.10. *Compare with the second Jackson q-Taylor formula* [9, (51), p.77], *which is obtained by letting $n \to \infty$.*

$$\begin{aligned} F(x \boxplus_q y) = \sum_{k=0}^{n-1} \frac{y^k}{\{k\}_q!} q^{\binom{k}{2}} D_q^k F(x) + \\ \int_{t=0}^{y} D_{q,t}^n \left[F(x \boxplus_q t)\right] \frac{(-t)^{n-1}}{\{n-1\}_q!} q^{\binom{n}{2}} \, d_q(t). \end{aligned} \tag{33}$$

Proof. We show that this is equivalent with (24). By putting $F(x) = x^m$ it would suffice to prove that

$$\sum_{n=1}^{m} \frac{y^n}{\{n\}_q!} q^{\binom{n}{2}} D_q^n x^m = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!} (-1)^{k+1} q^{\binom{k}{2}} D_{q,y}^k (x \boxplus_q y)^m. \tag{34}$$

This is equivalent to the formula

$$\begin{aligned} \sum_{n=1}^{m} \frac{y^n}{\{n\}_q!} \{m-n+1\}_{n,q} q^{\binom{n}{2}} x^{m-n} = \sum_{k=1}^{m} \frac{y^k}{\{k\}_q!} (-1)^{k+1} \{m-k+1\}_{k,q} \\ \sum_{l=0}^{m-k} \frac{\{m-k\}_q!}{\{m-k-l\}_q! \{l\}_q!} x^l y^{m-k-l} \\ \mathrm{QE}\left(k^2 - k + k(m-k-l) + \frac{(m-k-l)^2 - (m-k-l)}{2}\right). \end{aligned} \tag{35}$$

By equating the corresponding exponents for x and y, and thus putting $n = m - l$ we obtain

$$\begin{aligned} \frac{\{l+1\}_{m-l,q}}{\{n\}_q!} q^{\frac{m^2+l^2-2ml+l-m}{2}} = \sum_{k=1}^{m-l} \frac{(-1)^{k+1} \{m-k+1\}_{k,q}}{\{k\}_q!} \frac{\{m-k\}_q!}{\{m-k-l\}_q! \{l\}_q!} \\ \mathrm{QE}\left(k^2 - k + k(m-k-l) + \frac{(m^2+k^2+l^2-2mk-2ml+2kl) - (m-k-l)}{2}\right). \end{aligned} \tag{36}$$

After simplification we see that this is equivalent to (18) for the special case $u = 1$. □

References

[1] Al-Salam W. A.: q-Bernoulli numbers and polynomials. Math. Nachr. **17**, 239–260 (1959)

[2] Annaby, M. H. Mansour Z. S., q-Taylor and interpolation series for Jackson q-difference operators. J. Math. Anal. Appl. 344 , no. 1, 472–483 (2008). Math. Nachr 17, 239–260 (1959).

[3] Cigler J.: Operatormethoden für q-Identitäten. Monatshefte für Mathematik 88, 87-105 (1979)
[4] Ernst T: The history of q-calculus and a new method, Uppsala (2000)
[5] Ernst T: A new method for q-calculus, Uppsala dissertations (2002)
[6] Gauss C.F.: Werke 2 Göttingen (1876)
[7] Hahn W.: Beiträge zur Theorie der Heineschen Reihen. Mathematische Nachrichten2, 340-379 (1949)
[8] Jackson F.H.: A q-form of Taylor's theorem. Mess. Math. 38 , 62-64 (1909)
[9] Jackson F.H.: On basic double hypergeometric functions. Quart. J. Math., Oxford Ser. 13, 69–82 (1942)
[10] Nalli P.: Sopra un procedimento di calcolo analogo alla integrazione. (Italian) Rend. Circ. Mat. Palermo, Ser. I, 47, 337-374 (1923)
[11] Rothe H.A.: Systematisches Lehrbuch der Arithmetik Leipzig (1811)
[12] Schendel L.: Zur Theorie der Functionen. (x) [J] Borchardt J. LXXXIV. , 80-84 (1877)
[13] von Grüson J.F.: Neuer analytischer Lehrsats, Abhandlungen der Berliner Academie (1814-1815)
[14] Wallisser R.: Über ganze Funktionen, die in einer geometrischen Folge ganze Werte annehmen. Monatshefte Mathematik 100, , 329-335 (1985)
[15] Ward M.: A calculus of sequences. Amer. J. Math. 58, 255–266 (1936)

Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden Phone:+46(0)18261924 Fax:+46(0)18471 3201
E-mail address: thomas@math.uu.se