q- PASCAL AND q-BERNOULLI MATRICES, AN UMBRAL APPROACH

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Abstract A q-analogue $H_{n,q} \in Mat(n)(\mathbb{C}(q))$ of the Polya-Vein matrix is used to define the q-Pascal matrix. The Nalli–Ward–AlSalam (NWA) q-shift operator acting on polynomials is a commutative semigroup. The q-Cauchy-Vandermonde matrix generalizing Aceto-Trigiante is defined by the NWA q-shift operator. A new formula for a q-Cauchy-Vandermonde determinant with matrix elements equal to q-Ward numbers is found. The matrix form of the q-derivatives of the q-Bernoulli polynomials can be expressed in terms of the $H_{n,q}$. With the help of a new q-matrix multiplication certain special q-analogues of Aceto-Trigiante and Brawer-Pirovino are found. The q-Cauchy-Vandermonde matrix can be expressed in terms of the q-Bernoulli matrix. With the help of the Jackson-Hahn-Cigler (JHC) q-Bernoulli polynomials, the q-analogue of the Bernoulli complementary argument theorem is obtained. Analogous results for q-Euler polynomials are obtained. The q-Pascal matrix is factorized by the summation matrices and the so-called q-unit matrices.

1. Introduction

In this paper we are going to find q-analogues of matrix formulas from two pairs of authors: L. Aceto, & D. Trigiante [1], [2] and R. Brawer & M.Pirovino [5]. The umbral method of the author [11] is used to find natural q-analogues of the Pascal- and the Cauchy-Vandermonde matrices. The paper is organized as follows: In this chapter we give a general introduction.

In chapter two the first matrix calculations are made. A q-analogue $H_{n,q} \in Mat(n)(\mathbb{C}(q))$ of the Polya-Vein matrix is used to define the q-Pascal matrix $P_{n,q}$. In order to find proper q-analogues, sometimes many different q-analogues of the Pascal matrix are invented. The q-Cauchy-Vandermonde matrix generalizing [1] is defined by the NWA q-shift operator.

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Let $y(t)$ be a vector of length n. The following q-difference equation in $\mathbb{R}^n$ is of fundamental importance in this paper:

$$D_q y(t) = H_{n,q} y(t), y(0) = y_0, \ -\infty < t < \infty. \tag{1}$$

The solution of the matrix q-difference equation (1) can be expressed in terms of the $P_{n,q}$. The general solution of (1) is the q-Appell polynomial of degree ν and order m. The initial values are then the q-Bernoulli-, q-Euler- or q-Hermite numbers of order m etc. The initial value can also be the vector function e_0. Then the solution is the vector function $\xi(t)$. A slight modification of the initial value gives as solution the q-Cauchy matrix. This elegant property of Appell polynomials ($q = 1$) is summarized in Scaravelli [26].

The important q-matrix multiplication is introduced here, it will have many future applications. A new formula for a q-Cauchy determinant expressed as a product of q-Ward numbers is found, to this end certain q-Stirling numbers are introduced. The proof by induction uses Lagrange interpolation. The interested reader can find a historical introduction to these determinants in [10].

In chapter three the q-Bernoulli matrices are considered, i.e. matrix forms of q-Bernoulli polynomials. In the fourth chapter corresponding formulas for q-Euler matrices are given. The q-Cauchy-Vandermonde matrix can be expressed in terms of the q-Bernoulli matrix. The link is $L_{n,q}$, the definite q-integral of the q- Pascal matrix. There are two types of q-Bernoulli polynomials, NWA and Jackson-Hahn-Cigler (JHC), and the same for q-Euler polynomials. Elegant symmetry relations between the two types, NWA and JHC, for each case are found. Determinant formulas for the q-Bernoulli- and q-Euler numbers are given. With the help of the JHC q-Bernoulli polynomials, the q-analogue of the Bernoulli complementary argument theorem is obtained. Analogous results for q-Euler polynomials are obtained. In the fifth chapter, some more formulas for q-Pascal matrices are given. The summation matrix G and the difference matrix F are used to find q-analogues of [5]. In the formula for factorization of the q-Pascal matrix, certain q-unit matrices $I_{n,k,q}$ are used. Further formulas of the same character are expected. In an appendix some combinatorial formulas relevant to the text are given.

We only do formal computations, the convergence region in certain cases will have to be figured out by direct computation.

2. First matrix calculations

Definition 1. Matrix elements will always be denoted (i, j). Here i denotes the row and j denotes the column. The matrix elements range from 0 to $n-1$. Juxtaposition of matrices will always be interpreted as matrix multiplication. If z $n \times n$ matrix,

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{2}$$

If A and B are matrices of the same dimension, we define $A \oplus_q B$ as a matrix with matrix elements $A(i,j) \oplus_q B(i,j)$.

Theorem 2.1. *If A and B are commuting matrices,*

$$\mathrm{E}_q(A \oplus_q B) = \mathrm{E}_q(A)\mathrm{E}_q(B). \tag{3}$$

If a given matrix has been transformed to Jordan form $J = \Lambda + N$, with commuting terms Λ (diagonal) and N (nilpotent), and then computing $\mathrm{E}_q(\Lambda)$ and $\mathrm{E}_q(N)$, where the second series terminates, formula (3) gives a way of computing $\mathrm{E}_q(\Lambda \oplus_q N)$.

Definition 2. A q-analogue of the Polya-Vein matrix [1, (1), p. 232],[27, p. 278 (5)], [24, p. 257]. The $n \times n$ matrix $H_{n,q}$ is given by

$$\begin{aligned} &H_{n,q}(i, i-1) \equiv \{i\}_q,\ i = 1, \ldots, n-1, \\ &H_{n,q}(i,j) \equiv 0,\ j \neq i-1. \end{aligned} \tag{4}$$

We make the convention that all $n \times n$ matrices are denoted by first index n.

The matrix $H_{n,q}$ has the property

$$H_{n,q} e_i = \{i+1\}_q e_{i+1},\ i = 0, \ldots, n-1, \tag{5}$$

where $e_i,\ i = 0, \ldots, n-1$ denote the standard unit basis vectors in $\mathbb{R}^n$.

We immediately obtain

$$H_{n,q}^k e_i = \{i+1\}_{k,q} e_{i+k},\ k = 0, \ldots, n-1, \tag{6}$$

or equivalently a q-analogue of [27, p. 279].

$$H_{n,q}^k(i+k, i) = \{i+1\}_{k,q}. \tag{7}$$

According to our notation, $e_i = 0,\ i \geq n$ because $H_{n,q}^k = 0,\ k \geq n$.

Example 1.

$$H_{4,q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \{2\}_q & 0 & 0 \\ 0 & 0 & \{3\}_q & 0 \end{pmatrix} \tag{8}$$

$$H_{4,q}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \{1\}_{2,q} & 0 & 0 & 0 \\ 0 & \{2\}_{2,q} & 0 & 0 \end{pmatrix} \tag{9}$$

Definition 3. The matrices I_n, S_n, A_n and D_n [5, p. 13] are defined by

$$I_n \equiv \mathrm{diag}(1, 1, \ldots, 1) \tag{10}$$

$$S_n(i,j) \equiv \begin{cases} 1, & \text{if } j \leq i, \\ 0, & \text{if } j > i, \end{cases} \tag{11}$$

$$A_n(t)(i,j) \equiv \begin{cases} t^i, & \text{if } j = i, \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

$$D_n(i,i) \equiv 1 \text{ for all } i, \tag{13}$$

$$D_n(i+1,i) \equiv -1, \text{ for } i = 0, \ldots, n-2 \tag{14}$$

$$D_n(i,j) \equiv 0, \text{ if } j > i \text{ or } j < i-1 \tag{15}$$

We have

$$S_n = D_n^{-1}. \tag{16}$$

The unique solution of (1) is $y(t) = \mathrm{E}_q(H_{n,q}t)y_0$, where the q-exponential matrix $P_{n,q}(t) \equiv \mathrm{E}_q(H_{n,q}t)$ is given by the familiar expression

Definition 4.

$$P_{n,q}(t) \equiv \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!} H_{n,q}^k. \tag{17}$$

This is actually a finite series, whose matrix elements are polynomial in $H_{n,q}$ given by a q-analogue of [27, p. 278 (2)].

$$P_{n,q}(t)(i,j) = \sum_{k=0}^{n-1} \frac{t^k}{\{k\}_q!} H_{n,q}^k(i,j) = \binom{i}{j}_q t^{i-j}. \tag{18}$$

The following special case will often be used.

Definition 5. The q-Pascal matrix $P_{n,q}$ is given by

$$P_{n,q}(i,j) \equiv P_{n,q}(i,j)(1) = \binom{i}{j}_q, i,j = 0, \ldots, n-1. \tag{19}$$

Furthermore we have the following q-analogue of [1, p. 233 (7)], which follows from the fact that $P_{n,q}(t)$ is a q-exponential function.

$$P_{n,q}(s \oplus_q t) = P_{n,q}(s)P_{n,q}(t), \ \ s,t \in \mathbb{C}. \tag{20}$$

This implies

$$P_{n,q}^k = P_{n,q}(\overline{k}_q). \tag{21}$$

By (20) we obtain many combinatorial identities. Some of these are (122) and

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q = (\overline{2_q})^{i-j} \binom{i}{j}_q, \; i \geq j. \tag{22}$$

The q-Pascal matrix can also be expressed as a a q-analogue of [1, p. 233, (11)]

$$P_{n,q}(t) = A_n(t) P_{n,q}(A_n(t))^{-1}, \; t \neq 0. \tag{23}$$

In order to be able to write down certain q-matrix multiplication formulas, the following definition will be convenient.

Definition 6. Let A and B be two $n \times n$ matrices, with matrix index a_{ij} and b_{ij}, respectively. Then we define

$$AB_{f,q}(i,j) \equiv \sum_{m=0}^{n-1} a_{im} b_{mj} q^{f(m,i,j)}. \tag{24}$$

Whenever we use a q-matrix multiplication, we specify the corresponding function $f(m,i,j)$.

There is also a variant of the q-Pascal matrix.

Definition 7. The symmetric q-Pascal matrix $p_{n,q}$, a q-analogue of [1, (12), p. 233], [5, p. 17], is given by

$$p_{n,q}(i,j) \equiv \binom{i+j}{i}_q, i,j = 0, \ldots, n-1. \tag{25}$$

Theorem 2.2. *A q-analogue of* [2, p. 19]. *The matrix elements of the symmetric q-Pascal matrix are given by*

$$\sum_{k=0}^{\min(i,j)} \binom{i}{k}_q \binom{j}{k}_q q^{k^2} = \binom{i+j}{i}_q, i,j = 0, \ldots, n-1. \tag{26}$$

Proof. Use the first q-Vandermonde theorem. □

Corollary 2.3. *A q-analogue of* [1, p. 233 (12)], [2, p. 221, (12)]. *Let T denote matrix transposition. The symmetric q-Pascal matrix can be expressed as the q-matrix multiplication $P_{n,q}P_{n,q}^T$, with $f(m,i,j) = m^2$.*

We will need some more notation before we embark on the q-Cauchy matrix.

Definition 8. The Nalli-Ward-Alsalam q-shift operator [3, p. 242, 3.1] is given by

$$E(\oplus_q)(t^n) \equiv (t \oplus_q 1)^n \tag{27}$$

This can be generalized to

$$E(\oplus_q)^a(x^n) \equiv (x \oplus_q a)^n \tag{28}$$

The Jackson-Hahn-Cigler q-shift operator is given by

$$E(\boxplus_q)(t^n) \equiv (t \boxplus_q 1)^n \tag{29}$$

Definition 9. We define a set U, in which for every ordered pair of elements $a, b \in U$

$$E(\oplus_q)^a E(\oplus_q)^b(x^n) \equiv (x \oplus_q a \oplus_q b)^n. \tag{30}$$

Theorem 2.4. *The set U is a commutative semigroup (or monoid).*

Proof. Use the associativity and commutativity of $\oplus_q$. □

The q-difference operator is an infinitesimal generator of the semigroup.

The following abbreviation will be used.

$$\xi(t) \equiv (1, t, t^2, \ldots, t^{n-1})^T. \tag{31}$$

To save space in the following, we only write formulas for NWA (Ward) in certain cases.

The corresponding formulas for JHC (Jackson) follow from the next conversion table.

NWA	$\oplus_q$	(32)
JHC	$\boxplus_q$	

The following equations for NWA have JHC equivalents according to (32): (62), (64), (95).

Theorem 2.5. *The operator $E(\oplus_q)$ operating on $\xi(t)$ is equivalent to matrix multiplication with $P_{n,q}$.*

Definition 10. A q-analogue of [1, p. 234, (15)]. The q-Cauchy matrix is given by

$$\begin{aligned} W_{n,q}(t) &\equiv (\xi(t) \quad E(\oplus_q)\xi(t) \quad E(\oplus_q)^{\overline{2}_q}\xi(t) \quad \cdots \quad E(\oplus_q)^{\overline{n-1}_q}\xi(t)) \\ &\equiv (\xi(t) \quad (\xi(t \oplus_q 1) \quad (\xi(t \oplus_q \overline{2_q}) \quad \cdots \quad (\xi(t \oplus_q \overline{(n-1)_q})). \end{aligned} \tag{33}$$

Theorem 2.6. *The matrix elements of the q-Cauchy matrix are given by*

$$W_{n,q}(t)(i,j) = (t \oplus_q \overline{j_q})^i, i, j = 0, \ldots, n-1. \tag{34}$$

Theorem 2.7. *A q-analogue of* [1, p. 234]. *The vector function $\xi(t)$ satisfies (1) with $y_0 = e_0$. The q-Cauchy matrix satisfies (1).*

We are now going to find a new formula for a q-Cauchy-Vandermonde determinant expressed as a product of q-Ward numbers. To this end we first state a lemma. The proof by induction uses the following q-Stirling numbers.

Lemma 2.8. [14, p. 60] *If A and B are quadratic matrices, not necessarily of the same dimension, then*

$$\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = detA \; detB. \tag{35}$$

Definition 11. Define certain q-Stirling numbers $\{s_{n,k}(q)\}_{n,k=0}^{\infty} \in \mathbb{C}(q)$ by the following system of equations

$$\sum_{k=0}^{n} s_{n,k}(q)(\overline{i_q})^k = \delta_{i,n}\{n\}_q!, i = 0, 1, \ldots, n, \tag{36}$$

It follows that we get the following special value for these q-Stirling numbers:

$$s_{n,n}(q) = 1, \;\; s_{n,0}(q) = \delta_{0,n} \tag{37}$$

$$s_{2,1}(q) = -1, \;\; s_{3,1}(q) = \frac{1+q+2q^2}{1+q}, \tag{38}$$

$$s_{3,2}(q) = \frac{-2-2q-2q^2}{1+q}. \tag{39}$$

Theorem 2.9. *A Cauchy determinant for q-Ward numbers.*

$$det(W_{n,q}(0)) = \prod_{j=1}^{n-2} \{n-j\}_q^j \tag{40}$$

Proof. We use induction. The theorem is true for $n = 1$. Assume that it is true for $n-1$.

Then we have

$$det(W_{n,q}(0)) = \begin{vmatrix} det(W_{n-1,q}(0)) & A \\ B & (\overline{n-1_q})^{n-1} \end{vmatrix}, \tag{41}$$

where

$$A \equiv ((\overline{n-1_q})^0, (\overline{n-1_q})^1, \ldots, (\overline{n-1_q})^{n-2})^T, \tag{42}$$

$$B \equiv ((\overline{0_q})^{n-1}, (\overline{1_q})^{n-1}, \ldots, (\overline{n-2_q})^{n-1}) \tag{43}$$

By (35) it would suffice to prove that if we add $s_{n-1,0}(q)$ to row $0, s_{n-1,1}(q)$ to row $1 \ldots, s_{n-1,n-2}(q)$ to row $n-2$, with the following constraint:

$$(\overline{i_q})^{n-1} + \sum_{k=0}^{n-2} s_{n-1,k}(q)(\overline{i_q})^k = 0, i = 0, 1, \ldots, n-2, \tag{44}$$

then we would get for the matrix element $(n-1, n-1)$.

$$(\overline{n-1}_q)^{n-1} + \sum_{k=0}^{n-2} s_{n-1,k}(q)(\overline{n-1}_q)^k = \{n-1\}_q!. \tag{45}$$

The result now follows from (36), a kind of Lagrange interpolation in $\mathbb{C}(q)$. ☐

The matrix $P_{n,q}$ is triangular with the same characteristic polynomial as for $q = 1$: $P(\lambda) = (1-\lambda)^n$. This implies that the Frobenius matrix

$$F = W_{n,q}(0)^{-1} W_{n,q}(1) \tag{46}$$

is identical to the case $q = 1$ from [1, p. 235].

3. q-Bernoulli matrices and polynomials

In this chapter we will treat q-Bernoulli polynomials, and in the next chapter we will treat q-Euler polynomials. Both of these polynomials are special cases of q-Appell polynomials, which we will now define.

Definition 12. For every formal power series $f(t)$, the Φ_q polynomials of degree ν have the following generating function

$$f(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu,q}(x). \tag{47}$$

By putting $x = 0$, we have

$$f(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi_{\nu,q}, \tag{48}$$

where $\Phi_{\nu,q}$ is called a Φ_q number of degree ν.

Definition 13. There are two types of q-Bernoulli polynomials, called $\mathrm{B}_{\mathrm{NWA},\nu,q}(x)$, NWA q-Bernoulli polynomials, and $B_{\mathrm{JHC},\nu,q}(x)$, JHC q-Bernoulli polynomials. They are defined by the two generating functions

$$\frac{t}{(\mathrm{E}_q(t)-1)} \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{NWA,\nu,q}(x)}{\{\nu\}_q!}, \ |t| < 2\pi. \tag{49}$$

and

$$\frac{t}{(\mathrm{E}_{\frac{1}{q}}(t)-1)} \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{JHC,\nu,q}(x)}{\{\nu\}_q!}, \ |t| < 2\pi. \tag{50}$$

Definition 14. The Ward q-Bernoulli numbers [28, p. 265, 16.4], [3, p. 244, 4.1] are given by

$$B_{\text{NWA},n,q} \equiv B_{\text{NWA},n,q}(0). \tag{51}$$

The Jackson q-Bernoulli numbers are given by

$$B_{\text{JHC},n,q} \equiv B_{\text{JHC},n,q}(0). \tag{52}$$

The following table lists some of the first Ward q-Bernoulli numbers.

$n=0$	$n=1$	$n=2$	$n=3$
1	$-(1+q)^{-1}$	$q^2(\{3\}_q!)^{-1}$	$(1-q)q^3(\{2\}_q)^{-1}(\{4\}_q)^{-1}$

$n=4$
$q^4(1-q^2-2q^3-q^4+q^6)(\{2\}_q^2\{3\}_q\{5\}_q)^{-1}$

The following table lists some of the first Jackson q-Bernoulli numbers.

$n=0$	$n=1$	$n=2$	$n=3$
1	$-q(1+q)^{-1}$	$q^2(\{3\}_q!)^{-1}$	$(q^4-q^3)(\{2\}_q)^{-1}(\{4\}_q)^{-1}$

$n=4$
$q^4(1-q^2-2q^3-q^4+q^6)(\{2\}_q^2\{3\}_q\{5\}_q)^{-1}$

Theorem 3.1. *A symmetry theorem for q-Bernoulli numbers. For ν even,*

$$B_{\text{NWA},\nu,q} = B_{\text{JHC},\nu,q}. \tag{53}$$

For ν odd, $\nu > 1$,

$$B_{\text{NWA},\nu,q} = -B_{\text{JHC},\nu,q}. \tag{54}$$

Proof. Use the generating function (49) with $-t$ and $x = 0$, multiply with $\mathrm{E}_q(t)$ in denominator and numerator. Finally compare with the generating function (50). □

The recurrence formula for q-Bernoulli numbers can be written in the following matrix form for $n = 3$.

$$\begin{pmatrix} \{2\}_q & 0 & 0 \\ \{3\}_q & \{3\}_q & 0 \\ \{4\}_q & \binom{4}{2}_q & \{4\}_q \end{pmatrix} \begin{pmatrix} B_{\text{NWA},1,q} \\ B_{\text{NWA},2,q} \\ B_{\text{NWA},3,q} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}. \tag{55}$$

The general determinant formula for q-Bernoulli numbers follows from Cramer's rule, a q-analogue of [4, p. 131].

$$B_{\mathrm{NWA},n,q} = \frac{1}{\{n+1\}_q!}\begin{vmatrix} \{2\}_q & 0 & 0 & \dots & \dots & -1 \\ \{3\}_q & \{3\}_q & 0 & \dots & \dots & -1 \\ \{4\}_q & \binom{4}{2}_q & \binom{4}{3}_q & \dots & \dots & -1 \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ \binom{n+1}{1}_q & \binom{n+1}{2}_q & \binom{n+1}{3}_q & \dots & \binom{n+1}{n-1}_q & -1 \end{vmatrix}. \tag{56}$$

We will use the following vector forms for these polynomials corresponding to q-analogues of [1, p. 239].

$$b_{\mathrm{NWA},q}(x) \equiv (B_{\mathrm{NWA},0,q}(x), B_{\mathrm{NWA},1,q}(x), \dots, B_{\mathrm{NWA},n-1,q}(x))^T. \tag{57}$$

$$b_{\mathrm{JHC},q}(x) \equiv (B_{\mathrm{JHC},0,q}(x), B_{\mathrm{JHC},1,q}(x), \dots, B_{\mathrm{JHC},n-1,q}(x))^T. \tag{58}$$

The corresponding vector forms for numbers are

$$b_{\mathrm{NWA},q} \equiv (B_{\mathrm{NWA},0,q}, B_{\mathrm{NWA},1,q}, \dots, B_{\mathrm{NWA},n-1,q})^T. \tag{59}$$

$$b_{\mathrm{JHC},q} \equiv (B_{\mathrm{JHC},0,q}, B_{\mathrm{JHC},1,q}, \dots, B_{\mathrm{JHC},n-1,q})^T. \tag{60}$$

Definition 15. The q-integral is defined by

$$\int_0^a f(t,q)\, d_q(t) \equiv a(1-q)\sum_{n=0}^{\infty} f(aq^n,q)q^n,\ 0<|q|<1,\ a\in\mathbb{R}. \tag{61}$$

The q-Bernoulli polynomials obey the relations

$$D_q B_{\mathrm{NWA},i,q}(x) = \{i\}_q B_{\mathrm{NWA},i-1,q}(x). \tag{62}$$

$$\delta_{0,i} = \int_0^1 B_{\mathrm{NWA},i,q}(x)\, d_q(x). \tag{63}$$

These relations can be written in the following vector forms. The first one is our original q-difference equation (1).

$$D_q b_{\mathrm{NWA},q}(x) = H_{n,q} b_{\mathrm{NWA},q}(x). \tag{64}$$

$$e_0 = \int_0^1 b_{\mathrm{NWA},q}(x)\, d_q(x). \tag{65}$$

We will now introduce an important matrix, which turns out to form a link between the q-Cauchy matrix and a so-called q-Bernoulli matrix.

Definition 16. The matrix $L_{n,q}$ is defined by the following equivalent q-analogues of [1, p. 240].

$$L_{n,q} = \int_0^1 P_{n,q}(t)\, d_q(t) = \int_0^1 \sum_{k=0}^{n-1} \frac{t^k}{\{k\}_q!} H_{n,q}^k \, d_q(t) = \sum_{k=0}^{n-1} \frac{H_{n,q}^k}{\{k+1\}_q!}. \tag{66}$$

The related matrix $\widetilde{L_{n,q}}$ is given by

$$\widetilde{L_{n,q}} \equiv A_n(-1) L_{n,q} A_n(-1)^{-1}, \tag{67}$$

We have

$$\widetilde{L_{n,q}} = \sum_{k=0}^{n-1} \frac{(-H_{n,q})^k}{\{k+1\}_q!}. \tag{68}$$

Example 2.

$$L_{4,q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (\{2\}_q)^{-1} & 1 & 0 & 0 \\ (\{3\}_q)^{-1} & 1 & 1 & 0 \\ (\{4\}_q)^{-1} & 1 & \frac{\{3\}_q}{\{2\}_q} & 1 \end{pmatrix} \tag{69}$$

$$\widetilde{L_{4,q}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -(\{2\}_q)^{-1} & 1 & 0 & 0 \\ (\{3\}_q)^{-1} & -1 & 1 & 0 \\ -(\{4\}_q)^{-1} & 1 & -\frac{\{3\}_q}{\{2\}_q} & 1 \end{pmatrix} \tag{70}$$

By (1), (62) we have the following q-analogue of [1, p. 239, (37)].

$$b_{\mathrm{NWA},q}(x) = P_{n,q}(x) b_{\mathrm{NWA},q}(0). \tag{71}$$

From this and (63) we obtain

$$\int_0^1 P_{n,q}(x)\, d_q(x) b_{\mathrm{NWA},q}(0) = L_{n,q} b(0) = e_0. \tag{72}$$

Theorem 3.2. *A q-analogue of* [1, p. 240, (40)]. *Let the q-matrix multiplication be defined by $f(m,i,j) = q^{\binom{m-j}{2}}$.*

The matrix $M_{n,q}$ is obtained by multiplying all matrix elements in $L_{n,q}$, except the diagonal ones, by q.

Then

$$M_{n,q} = P_{n,q} \widetilde{L_{n,q}}_{f,q}. \tag{73}$$

Proof. By equating the matrix indices of both sides, it will suffice to prove that when $k > 0$

$$\begin{aligned}&\frac{q\{j+1\}_{k,q}}{\{k+1\}_q!} = \sum_{m=0}^{n-1} \frac{(-1)^{m-j} q^{\binom{m-j}{2}} \{j+1\}_{m,q}}{\{m+1\}_q!} \binom{j+k}{m}_q = \\ &\sum_{m=0}^{k} \frac{(-1)^m q^{\binom{m}{2}} (1-q) \langle 1;q\rangle_{j+k}}{\langle 1;q\rangle_{1+m} \langle 1;q\rangle_j \langle 1;q\rangle_{k-m}},\end{aligned} \tag{74}$$

which implies

$$q = \sum_{m=0}^{k} (-1)^m \binom{k+1}{1+m}_q q^{\binom{m}{2}}, \tag{75}$$

which is equivalent to the following famous equation of Gauss:

$$\sum_{n=0}^{m} (-1)^n \binom{m}{n}_q q^{\binom{n}{2}} a^n = (a;q)_m, \tag{76}$$

where

$$(a;q)_n \equiv \begin{cases} 1, & n = 0; \\ \prod_{m=0}^{n-1} (1 - aq^m), & n = 1, 2, \ldots, \end{cases} \tag{77}$$

□

The matrix $L_{n,q}^{-1}$ is also interesting. It can be expressed as

$$L_{n,q}^{-1} = \sum_{k=0}^{n-1} \frac{B_{\mathrm{NWA},k,q}}{\{k\}_q!} (H_{n,q})^k. \tag{78}$$

Theorem 3.3. *A q-analogue of* [1, p. 240, (42)], [2, p. 221, (10)].

$$b_{\mathrm{NWA},q}(x) = L_{n,q}^{-1} \xi(x). \tag{79}$$

Proof.

$$\begin{aligned}&b_{\mathrm{NWA},q}(x) \overset{\text{by}(71),(72)}{=} P_{n,q}(x) L_{n,q}^{-1} e_0 = L_{n,q}^{-1} P_{n,q}(x) e_0 \overset{\text{by}(17)}{=} L_{n,q}^{-1} \sum_{k=0}^{n-1} \frac{t^k}{\{k\}_q!} H_{n,q}^k e_0 \\ &\overset{\text{by}(6)}{=} L_{n,q}^{-1} \sum_{k=0}^{n-1} t^k e_k \overset{\text{by}(31)}{=} L_{n,q}^{-1} \xi(x).\end{aligned} \tag{80}$$

□

Let us now introduce the q-Bernoulli matrix

$$\overline{B}_{\mathrm{NWA},n,q}(x) \equiv (b_{\mathrm{NWA},q}(x) \;\; E(\oplus_q)b_{\mathrm{NWA},q}(x) \;\; \cdots \;\; E(\oplus_q)^{n-1}b_{\mathrm{NWA},q}(x)). \tag{81}$$

This implies that the q-Cauchy matrix can be written as the following q-analogue of [1, p. 241, (46)].

$$W_{n,q}(x) = L_{n,q}\overline{B}_{\mathrm{NWA},n,q}(x). \tag{82}$$

We also have the following q-analogue of [2, p. 221]

$$P_{n,q} = I_n + H_{n,q}L_{n,q}. \tag{83}$$

The q-analogue of the Bernoulli complementary argument theorem [11, p. 51, (3.134)] can be written in the matrix form

$$b_{\mathrm{JHC},q}(x) = A_n(-1)b_{\mathrm{NWA},q}(1 \ominus_q x). \tag{84}$$

4. q-Euler matrices and polynomials

There are two types of q-Euler polynomials, called $\mathrm{F}_{\mathrm{NWA},\nu,q}(x)$, NWA q-Euler polynomials, and $F_{\mathrm{JHC},\nu,q}(x)$, JHC q-Euler polynomials. They are defined by the two generating functions

$$\frac{2\mathrm{E}_q(xt)}{(\mathrm{E}_q(t)+1)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} F_{\mathrm{NWA},\nu,q}(x), \;\; |t| < \pi. \tag{85}$$

and

$$\frac{2\mathrm{E}_q(xt)}{(\mathrm{E}_{\frac{1}{q}}(t)+1)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} F_{\mathrm{JHC},\nu,q}(x), \;\; |t| < \pi. \tag{86}$$

We will use the following notation for q-Euler numbers:

$$F_{\mathrm{NWA},\nu,q} \equiv F_{\mathrm{NWA},\nu,q}(0), \tag{87}$$

and the same for JHC. The following table lists some of the first q-Euler numbers $F_{\mathrm{NWA},n,q}$.

$n=0$	$n=1$	$n=2$	$n=3$
1	-2^{-1}	$2^{-2}(-1+q)$	$2^{-3}(-1+2q+2q^2-q^3)$

$n=4$
$2^{-4}(q-1)\{3\}_q!(q^2-4q+1)$

By an elementary argument for the generating function we can prove the following symmetry theorem:

Theorem 4.1. *For ν even,*

$$F_{\mathrm{NWA},\nu,q} = -F_{\mathrm{JHC},\nu,q}. \tag{88}$$

For ν odd,

$$F_{\mathrm{NWA},\nu,q} = F_{\mathrm{JHC},\nu,q}. \tag{89}$$

Proof. Multiply the generating function (85) by $\mathrm{E}_q(t)$ for $x = 0$, to obtain

$$(-1)^\nu F_{\mathrm{JHC},\nu,q} = \sum_{k=0}^{\nu} \binom{\nu}{k}_q F_{\mathrm{NWA},\nu-k,q}. \tag{90}$$

Finally compare with the generating function (86). □

The recurrence formula for q-Euler numbers can be written in the following matrix form for $n = 4$.

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ \{2\}_q & 2 & 0 & 0 \\ \{3\}_q & \{3\}_q & 2 & 0 \\ \{4\}_q & \binom{4}{2}_q & \{4\}_q & 2 \end{pmatrix} \begin{pmatrix} F_{\mathrm{NWA},1,q} \\ F_{\mathrm{NWA},2,q} \\ F_{\mathrm{NWA},3,q} \\ F_{\mathrm{NWA},4,q} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}. \tag{91}$$

The general determinant formula for q-Euler numbers follows from Cramer's rule.

$$F_{\mathrm{NWA},n,q} = \frac{1}{2^n} \begin{vmatrix} 2 & 0 & 0 & \dots & 0 & -1 \\ \{2\}_q & 2 & 0 & \dots & 0 & -1 \\ \{3\}_q & \{3\}_q & 2 & \dots & 0 & -1 \\ \{4\}_q & \binom{4}{2}_q & \binom{4}{3}_q & 2 & \dots & -1 \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ \binom{n}{1}_q & \binom{n}{2}_q & \binom{n}{3}_q & \dots & \binom{n}{n-1}_q & -1 \end{vmatrix}. \tag{92}$$

We will use the following vector forms for these polynomials.

$$f_{\mathrm{NWA},q}(x) \equiv (F_{\mathrm{NWA},0,q}(x), F_{\mathrm{NWA},1,q}(x), \dots, F_{\mathrm{NWA},n-1,q}(x))^T. \tag{93}$$

$$f_{\mathrm{JHC},q}(x) \equiv (F_{\mathrm{JHC},0,q}(x), F_{\mathrm{JHC},1,q}(x), \dots, F_{\mathrm{JHC},n-1,q}(x))^T. \tag{94}$$

Let the operators $\nabla_{\mathrm{NWA},q}$ and $\nabla_{\mathrm{JHC},q}$ be defined by

$$\nabla_{\mathrm{NWA},q} \equiv \frac{E(\oplus_q) + I}{2}. \tag{95}$$

Then

$$\nabla_{\mathrm{NWA},q} f_{\mathrm{NWA},q}(x) = \xi(x). \tag{96}$$

Theorem 4.2. *The matrix $2(P_{n,q}+I_n)^{-1}$ transforms the basis of powers into the basis of q-Euler polynomials, a q-analogue of* [2, p. 223, (22).].

$$f_{\mathrm{NWA},q}(x) = 2(P_{n,q} + I_n)^{-1} \xi(x). \tag{97}$$

Proof. By (96)

$$f_{\mathrm{NWA},q}(x \oplus_q 1) + f_{\mathrm{NWA},q}(x) = 2\xi(x). \tag{98}$$

The LHS can be written as

$$[P_{n,q}(x \oplus_q 1) + P_{n,q}(x)]\, f_{\mathrm{NWA},q} = (P_{n,q} + I_n)P_{n,q}(x)f_{\mathrm{NWA},q}. \tag{99}$$

□

Corollary 4.3. *A connection between q-Bernoulli matrices and q-Euler matrices, a q-analogue of* [2, p. 223, (23).].

$$f_{\mathrm{NWA},q}(x) = 2(P_{n,q} + I_n)^{-1}L_{n,q}b_{\mathrm{NWA},q}(x). \tag{100}$$

The q-analogue of the Euler complementary argument theorem [11, p. 51, (3.135)] can be written in the matrix form

$$f_{\mathrm{JHC},q}(x) = A_n(-1)f_{\mathrm{NWA},q}(1 \ominus_q x). \tag{101}$$

This implies the following q-analogue of [2, p. 223].

$$f_{\mathrm{NWA},q}(x) - 2\xi(x) = -A_n(-1)f_{\mathrm{JHC},q}(-x). \tag{102}$$

5. More formulas for q-Pascal matrices

We now turn to certain variations of q-Pascal matrices with the aim of expanding $P_{n,q}$ as a product of lower triangular matrices in the spirit of Brawer & Pirovino [5].

Definition 17. The matrix $\overline{P_{n,q}}$ is defined by

$$\overline{P_{n,q}}(i,j) \equiv \binom{i-1}{j-1}_q q^{i-j}, i,j = 1,\ldots,n-1, \tag{103}$$

$$\overline{P_{n,q}}(i,j) \equiv \delta_{i,j}, i \text{ and } j = 0 \text{ or } 1. \tag{104}$$

Let S_n be given by (11), and let D_n be given by (13)–(15). The summation matrix $G_{n,k}$ and the difference matrix $F_{n,k}$ are defined by [5, p. 14-15]

$$\mathrm{G}_{n,k} \equiv \begin{bmatrix} I_{n-k} & 0^T \\ 0 & S_k \end{bmatrix}, G_{n,n} \equiv S_n$$
$$F_{n,k} \equiv \begin{bmatrix} I_{n-k} & 0^T \\ 0 & D_k \end{bmatrix}, F_{n,n} \equiv D_n,\ k = 0,\ldots,n-1. \tag{105}$$

Then we have

$$F_{n,k} = G_{n,k}^{-1}. \tag{106}$$

The following lemma enables a step by step proof of (114).

Lemma 5.1. *A q-analogue of* [5, p. 14].

$$S_n\overline{P_{n,q}} = P_{n,q},\ n \geq 1. \tag{107}$$

Proof. For $n = 1$, $\overline{P_{n,q}} = I_n$ and $S_n = P_{n,q}$. Let $n > 1$. The matrix element

$$S_n\overline{P_{n,q}}(i,j) = \sum_{l=j}^{i} \binom{l-1}{j-1}_q q^{l-j} = \binom{i}{j}_q = P_{n,q}(i,j), j \geq 1. \tag{108}$$

For $j = 0$, $S_n\overline{P_{n,q}}(i,0) = 1 = P_{n,q}(i,0)$. □

Definition 18. Let the three lower triangular matrices $I_{n,k,q}$, $E_{n,k,q}$ and $\widetilde{P_{n,q}}$ be given by

$$I_{n,k,q}(i,i) \equiv 1, i = 0,\ldots,n-1,\ I_{n,k,q}(i+1,i) \equiv q^i - 1,\ i = n-k\ldots,n-2. \tag{109}$$

$I_{n,k,q}(i,j) \equiv 0$ for other i, j.

$$E_{n,k,q}(i,i) \equiv 1, i = 0,\ldots,n-1,\quad E_{n,k,q}(i,j) \equiv \langle j+1;q\rangle_{i-j},\ i > j. \tag{110}$$

$$\widetilde{P_{n,q}}(i,j) \equiv \binom{i}{j}_q q^{i-j}. \tag{111}$$

The matrices $I_{n,k,q}$ and $E_{n,k,q}$ are inverse to each other. We call $I_{n,k,q}$ the q-unit matrix.

Lemma 5.2. *Let*

$$I_{n,q} \equiv I_{n,n,q},\quad E_{n,q} \equiv E_{n,n,q}. \tag{112}$$

Then

$$\widetilde{P_{n,q}} = I_{n,q}P_{n,q}. \tag{113}$$

Theorem 5.3. *A q-analogue of* [5, p. 15 (1)]. *If $n > 3$ the q-Pascal matrix $P_{n,q}$ can be factorized by the summation matrices and by the q-unit matrices as*

$$P_{n,q} = \prod_{k=n}^{2} G_{n,k}I_{n,k-1,q}, \tag{114}$$

where the product is taken in decreasing order of k.

Theorem 5.4. *A q-analogue of* [5, p. 15 (2)]. *The inverse of the q-Pascal matrix is given by*

$$P_{n,q}^{-1} = F_{n,2}\prod_{k=3}^{n} E_{n,k-1,q}F_{n,k}. \tag{115}$$

Definition 19. Let $P_{n,q}*$ be defined by $P_{n,q}*(i,j) \equiv (-1)^{i+j}P_{n,q}(i,j)$ and

$$\overline{P_{n,q}}* \equiv \begin{bmatrix} 1 & 0^T \\ 0 & P_{n-1,q}* \end{bmatrix}. \tag{116}$$

Lemma 5.5. *A q-analogue of* [5, p. 15 Lemma 2]. *We are going to use a q-matrix multiplication for the following equation. This time we will describe the function $f(m,i,j)$ verbally as follows. Because of the sparse structure of the matrix D_n, only two terms survive for each fixed i,j. These terms take values of m in consequtive order. For these nonzero terms, $f(m,i,j)$ takes the value 0 for the lowest m, and $f(m,i,j) = m-1$ for for the next and highest value of m. Then*

$$\overline{P_{n,q}} * \; D_n = P_{n,q} * . \tag{117}$$

Proof. Use the q-Pascal identity. □

6. Appendix

Some of the following q-analogues of Riordan [25] *have been used in the preceding proofs. A q-analogue of* [25, p. 6]

$$\binom{n}{m}_q = \sum_{k=0}^{M}\binom{n-1-k}{m-k}_q q^{m-k}, M = \min(m, n-1). \tag{118}$$

A q-analogue of [25, p. 8, (4)]

$$\begin{gathered}\binom{n-p}{m}_q = \sum_{k=0}^{M}\binom{-p}{k}_q\binom{n}{m-k}_q q^{(m-k)(-p-k)} = \\ \sum_{k=0}^{M}\binom{p+k-1}{k}_q\binom{n}{m-k}_q q^{(m-k)(-p-k)-pk-\binom{p}{2}}(-1)^k.\end{gathered} \tag{119}$$

Theorem 6.1. *The inverse of the q-Pascal matrix is given by*

$$P_{n,q}^{-1} = \mathrm{E}_{\frac{1}{q}}(-H_{n,q}), \tag{120}$$

or equivalently

$$P_{n,q}^{-1}(i,j) = (-1)^{i-j}\binom{i}{j}_q q^{\binom{i-j}{2}}, i,j = 0,\ldots,n-1. \tag{121}$$

Corollary 6.2. *A q-analogue of* [1, p. 234].

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q (-1)^{k-j} q^{\binom{k-j}{2}} = \delta_{i,j}. \tag{122}$$

This can be compared with the following inverse relation obtained by the operator $\triangle_{\mathrm{CG},q}$.

Theorem 6.3. *A q-analogue of* [20, p. 133].

$$\binom{x+a}{m-n}_q q^{n(x+a+n-m)} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}_q \binom{x+a+k}{m}_q q^{\binom{n-k}{2}} \tag{123}$$

Proof. We will use the following operator by Carlitz and Gould.

$$\triangle_{\mathrm{CG},q} f(x) \equiv f(x+1) - f(x),\ \triangle_{\mathrm{CG},q}^{n+1} f(x) \equiv \triangle_{\mathrm{CG},q}^{n} f(x+1) - q^n \triangle_{\mathrm{CG},q}^{n} f(x). \tag{124}$$

Now use $\triangle_{\mathrm{CG},q}^{n} \binom{x+a}{m}_q = \binom{x+a}{m-n}_q q^{n(x+a+n-m)}$ □

7. Conclusion

We are only at the beginning of this subject, some possible paths to continue are to study further q-Appell polynomials or q-deformed matrix groups. Because of the close affinity to combinatorial identities and combinatorics, the q-matrix multiplication introduced here will have many applications in the theory of inverse relations from Riordan [25].

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