SOME RESULTS FOR q–FUNCTIONS OF MANY VARIABLES III

THOMAS ERNST

Abstract. We find four q-analogues of general reduction formulas from Buschman and Srivastava [3] together with some special cases, e.g. q-analogues of reduction formulas for Appell- and Kampé de Fériet functions. A proper q-analogue of the notation $\triangle(l;\lambda)$ by MacRobert [15], Meijer and Srivastava [25] is given, and the definition of q-hypergeometric series is generalized accordingly. Some further q-analogues of Srivastava from [28],[29] and [30] are found. We introduce the concepts of balanced and very-well-poised double q-hypergeometric series and we find double q-analogues of Srivastava's summation formulas from [22].

1. Introduction

The purpose of this paper is to continue the study of q–Appell-, q–Lauricella- and q–Kampé de Fériet functions from [5]. The convergence properties of these q-series is a rather delicate problem, which will have to be relegated until later. Instead we concentrate on a multitude of double series formulas. This paper is organized as follows: In this section we give a general introduction. In section 2, we give a slightly improved version of the q–Kampé de Fériet function compared with [5], stating exactly the convergence region for the case $q = 1$ and $n = 2$, due to Srivastava. The important notation $\triangle(l;\lambda)$ of MacRobert and Srivastava for a certain array of l parameters is given its proper q-analogue with the aid of a generalized tilde operator; in this paper we only consider the cases $l = 2, 3$, but a general definition is given. This $\triangle$-operator has a very long history in India and we will come back to this in later papers.

Buschman and Srivastava [3] have proved a great number of double series identities with general terms. We will find q-analogues of most of these formulas like in [5]; the method of proof will be similar except that we now use the q-Dixon- and q-Watson summation formulas. Some of the obtained formulas are symmetric in two variables, just as

Date: June 15, 2010.

[0]2010 Mathematics Subject Classification: Primary 33C65; Secondary 33D70

in the undeformed case. We pick out a form of these formulas, which converges nicely for small values of x. A list of the different formulas and q-analogues in various journals and books is given for better orientation. In section 2.4 we apply the Buschman-Srivastava formulas to find q-analogues of reduction formulas for Appell and Kampé de Fériet functions. In section 3 we further investigate q-analogues of Srivastava. The important concepts balanced and very-well-poised double q-hypergeometric series are introduced in section 3.1. In the last section we give a systematic collection of double q-analogues of summation formulas from [22], which are exemplified by q-analogues of [17]. The great difference between sec. 2 and 4 is that in section 2 the $\triangle$ operator appears only in the Heine function, whereas in section 4, the $\triangle$ operator appears in the q–Kampé de Fériet function.

We start with a new definition, other definitions are found in the references.

Definition 1. A q-analogue of a notation due to Thomas MacRobert (1884 - 1962) [15, p. 135] and Srivastava [25]. This notation was also often used for the Meijer G-function and the Fox H-function ($q = 1$).

$$\langle\triangle(q;l;\lambda);q\rangle_n \equiv \prod_{m=0}^{l-1}\langle\frac{\lambda+m}{l};q\rangle_n \times_l \langle\frac{\widetilde{\lambda+m}}{l};q\rangle_n. \tag{1}$$

When λ is a vector, we mean the corresponding product of vector elements. When λ is replaced by a sequence of numbers separated by commas, we mean the corresponding product as in the case of q-shifted factorials. The last factor in (1) corresponds to l^{nl}.

2. q–analogues of Buschman-Srivastava

2.1. **Definitions.** We will give a definition reminding of [11], which allows easy confluence to diminish the dimensions in (3) and (4), and has the advantage of beeing symmertic in the variables. Furthermore, q is allowed to be a vector and the full machinery of tilde operators and q–additions will be used.

In the following three definitions we put

$$\widehat{a} \equiv a \vee \tilde{a} \vee \widetilde{\tfrac{m}{n}a} \vee_k \widetilde{a} \vee \triangle(q;l;\lambda). \tag{2}$$

The first definition is a q–analogue of [26, (24), p. 38], in the spirit of Srivastava. The second definition is a q–analogue of [26, (24), p. 38] with the restraint [26, (29), p. 38], due to Karlsson. It will be clear from the context which of the definitions we use.

Definition 2. Let

$$(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)$$

have dimensions

$$A,B,G_i,H_i,A',B',G'_i,H'_i.$$

Let

$$1+B+B'+H_i+H'_i-A-A'-G_i-G'_i\geq 0, i=1,\ldots,n.$$

Then the generalized q–Kampé de Fériet function is defined by

(3)

$$\Phi^{A+A':G_1+G'_1;\ldots;G_n+G'_n}_{B+B':H_1+H'_1;\ldots;H_n+H'_n}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g'_1);\ldots;(g'_n)\\(b'):(h'_1);\ldots;(h'_n)\end{array}\right]\equiv$$

$$\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle 1;q_j\rangle_{m_j})}\times$$

$$(-1)^{\sum_{j=1}^n m_j(1+H_j+H'_j-G_j-G'_j+B+B'-A-A')}\times$$

$$\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n\mathrm{QE}\left((1+H_j+H'_j-G_j-G'_j)\binom{m_j}{2},q_j\right).$$

We assume that no factors in the denominator are zero. We assume that $(a')(q_0,m),(g'_j)(q_j,m_j),(b')(q_0,m),(h'_j)(q_j,m_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k,(s(k);q)_k$ or $\mathrm{QE}\left(f(\vec{m})\right)$.

Definition 3. Let

$$(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)$$

have dimensions

$$A,B,G,H,A',B',G',H'.$$

Let

$$1+B+B'+H+H'-A-A'-G-G'\geq 0.$$

Then the generalized q–Kampé de Fériet function is defined by

(4)

$$\Phi^{A+A':G+G'}_{B+B':H+H'}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g'_1);\ldots;(g'_n)\\(b'):(h'_1);\ldots;(h'_n)\end{array}\right]\equiv$$

$$\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle 1;q_j\rangle_{m_j})}\times$$

$$(-1)^{\sum_{j=1}^n m_j(1+H+H'-G-G'+B+B'-A-A')}\times$$

$$\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n\mathrm{QE}\left((1+H+H'-G-G')\binom{m_j}{2},q_j\right).$$

We assume that no factors in the denominator are zero. We assume that $(a')(q_0,m),(g'_j)(q_j,m_j),(b')(q_0,m),(h'_j)(q_j,m_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k,(s(k);q)_k$ or $\mathrm{QE}\left(f(\vec{m})\right)$.

Remark 1. The convergence region of the above series is extremely difficult to compute. Consider the function (3) for $q=1,n=2$ [29, p. 16]. When

$$A+A'+G_i+G'_i<B+B'+H_i+H'_i+1,\ i=1,2, \tag{5}$$

the series always converges. When

$$A+A'+G_i+G'_i=B+B'+H_i+H'_i+1,\ i=1,2, \tag{6}$$

the series converges for

$$\begin{cases}|x_1|^{\frac{1}{A+A'-B-B'}}+|x_2|^{\frac{1}{A+A'-B-B'}}<1,\ \text{if } A+A'>B+B'\\ \max\{|x_1|,|x_2|\}<1,\ \text{if } A+A'\leq B+B'\end{cases}. \tag{7}$$

Consider the function (4) for $q=1,n=2$. When

$$A+A'+G+G'<B+B'+H+H'+1, \tag{8}$$

the series always converges. When

$$A+A'+G+G'=B+B'+H+H'+1, \tag{9}$$

the series converges for

$$\begin{cases}|x_1|^{\frac{1}{A+A'-B-B'}}+|x_2|^{\frac{1}{A+A'-B-B'}}<1,\ \text{if } A+A'>B+B'\\ \max\{|x_1|,|x_2|\}<1,\ \text{if } A+A'\leq B+B'\end{cases}. \tag{10}$$

Definition 4. Generalizing Heine's series we shall define a q-hypergeometric series by

$$
(11)\quad \begin{aligned}
&{}_{p+p'}\phi_{r+r'}\left[\begin{matrix}\hat{a}_1,\ldots,\hat{a}_p\\ \hat{b}_1,\ldots,\hat{b}_r\end{matrix}\ |q;z||\ \begin{matrix}\prod_i f_i(k)\\ \prod_j g_j(k)\end{matrix}\right]\equiv\\
&\sum_{k=0}^{\infty}\frac{\langle\hat{a}_1;q\rangle_k\ldots\langle\hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k\ldots\langle\hat{b}_r;q\rangle_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'} z^k\frac{\prod_i f_i(k)}{\prod_j g_j(k)}.
\end{aligned}
$$

We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle\widehat{a(k)};q\rangle_k$ or $(s(k);q)_k$ respectively. In case of $\triangle(q;l;\lambda)$ the index is adapted accordingly. When we have a sequence of elements a_i we can denote them by (A).

2.2. **Two lemmata.** In the following three proofs we will use the finite q–Dixon-Schafheitlin theorem.

Theorem 2.1. [5, p. 210 (39)]

$$
(12)\quad \begin{cases}
{}_4\phi_3\left[\begin{matrix}-2k,b,c,\widetilde{1-k}\\ 1-2k-b,1-2k-c,\widetilde{-k}\end{matrix}\ |q;q^{1-k-b-c}\right]\equiv\\
\sum_{j=0}^{2k}\binom{2k}{j}_q\frac{\langle b,c,\widetilde{1-k};q\rangle_j(-1)^j\,\mathrm{QE}(\binom{j}{2}+j(1-3k-b-c))}{\langle 1-2k-b,1-2k-c,\widetilde{-k};q\rangle_j}\\
=\frac{\langle 1-2k-b-c,\widetilde{1-k},\widetilde{b};q\rangle_k}{\langle 1-2k-c,\widetilde{1-k-b},b+k;q\rangle_k}\langle\frac{1}{2};q^2\rangle_k;\\
{}_4\phi_3\left[\begin{matrix}-k,b,c,\widetilde{1-\frac{k}{2}}\\ 1-k-b,1-k-c,\widetilde{-\frac{k}{2}}\end{matrix}\ |q;q^{1-\frac{k}{2}-b-c}\right]=0,k\text{ odd.}
\end{cases}
$$

In another proof we will use

Theorem 2.2. *A q–analogue of the Schafheitlin-Watson formula* [8, p. 170 (43)] *:*

(13)

$$
{}_4\phi_3\left[\begin{matrix}\frac{c}{2},\widetilde{\frac{c}{2}},a,-N\\ \frac{-N+1+a}{2},\widetilde{\frac{-N+1+a}{2}},c\end{matrix}\ |q;q\right]=\begin{cases}=\frac{\langle\frac{1}{2},\frac{1+c-a}{2};q^2\rangle_{\frac{N}{2}}}{\langle\frac{1-a}{2},\frac{1+c}{2};q^2\rangle_{\frac{N}{2}}}, & \textit{if } N \textit{ even;}\\ 0, & \textit{if } N \textit{ odd.}\end{cases}
$$

2.3. **Four general double sums.** The Buschman-Srivastava paper [3] was a landmark for the studies of multiple q-hypergeometric series. Some of these formulas had previously been published by Shanker and Saran [19]. Srivastava and Jain [27] have found q-analogues of some of these formulas, some of which are included in the book [10]. The following table summarizes the connection between the various formulas and the method of proof. In the table below the first four references are in chronological order.

[19]	[23]	[3]	[31]	Proof	Equation no.
–	(4)	3.2	33	q-Vandermonde	[5](62), (66)
–	(5)	3.7	49	finite Bailey-Daum	[5](68)
–	(17)	3.3	44	finite Bailey-Daum	[5](81)
b p. 10	–	2.7, 3.4	46	finite q-Dixon	(14)
c p.10	–	2.8, 3.6	48	finite q-Dixon	(16)
–	–	2.9, 3.8	50	finite q-Dixon	(18)
a p.10	–	3.10, 3.5	47	q-Watson	(20)

We are now going to find a number of general double sums. Since the convergence problem is rather delicate, we try to choose the most proper form with respect to an arbitrary q-power. Sometimes we add this q-power afterwards. In the following, a statement like $a \neq k$ will mean $a \neq k, k \in \mathbb{N}$. Note that the formulas (16), (18) and (20) are symmetric in two variables.

Theorem 2.3. *A q-analogue of Buschman, Srivastava* [3, p. 437 (2.7)].

$$\begin{aligned}&\sum_{m,n} \frac{C_{m+n} x^{m+n} (-1)^n \langle g; q\rangle_m \langle g, \widetilde{1-\tfrac{m+n}{2}}; q\rangle_n \mathrm{QE}\left(-\tfrac{n}{2} - \tfrac{nm}{2}\right)}{\langle 1, h; q\rangle_m \langle 1, h, \widetilde{-\tfrac{m+n}{2}}; q\rangle_n} \\ &= \sum_{N=0}^{\infty} \frac{C_{2N} x^{2N} \langle g, h-g, \widetilde{1-N}; q\rangle_N}{\langle 1, \widetilde{1}, h, \tfrac{h}{2}, \widetilde{\tfrac{h}{2}}, \tfrac{h+1}{2}, \widetilde{\tfrac{h+1}{2}}; q\rangle_N} q^{\binom{N}{2} + Ng}, \ -h \neq k.\end{aligned} \tag{14}$$

Proof.

(15)

$$LHS=\sum_{N,n}\frac{C_Nx^N(-1)^n\langle g;q\rangle_{N-n}\langle g,\widetilde{1-\frac{N}{2}};q\rangle_n\mathrm{QE}\left(\binom{n}{2}-\frac{nN}{2}\right)}{\langle 1,h;q\rangle_{N-n}\langle 1,h,\widetilde{-\frac{N}{2}};q\rangle_n}=$$

$$\sum_{N=0}\frac{C_Nx^Nq^{N(g-h)}\langle -g+1-N;q\rangle_N}{\langle 1,-h+1-N;q\rangle_N}$$

$$\sum_{n}\frac{(-1)^n\binom{N}{n}_q\langle g,-h+1-N,\widetilde{1-\frac{N}{2}};q\rangle_n\mathrm{QE}\left(\binom{n}{2}+n(h-g-\frac{N}{2})\right)}{\langle h,-g+1-N,\widetilde{-\frac{N}{2}};q\rangle_n}$$

$$\overset{\text{by}(12)}{=}\sum_{N=0}\frac{C_{2N}x^{2N}q^{2N(g-h)}\langle -g+1-2N;q\rangle_{2N}}{\langle 1,-h+1-2N;q\rangle_{2N}}\Gamma_q\begin{bmatrix}1-2N-g,h,1-N,h-g+N\\1-2N,h-g,1-N-g,h+N\end{bmatrix}$$

$$=\sum_{N=0}\frac{C_{2N}x^{2N}\langle h-g,\widetilde{1-N},\widetilde{g};q\rangle_N\langle -g+1-2N;q\rangle_{2N}\langle\frac{1}{2};q^2\rangle_N\mathrm{QE}\left(\binom{2N}{2}+2Ng\right)}{\langle h,\frac{h}{2},\widetilde{\frac{h}{2}},\frac{h+1}{2},\widetilde{\frac{h+1}{2}},N+g,1\widetilde{-N}-g;q\rangle_N\langle 1;q\rangle_{2N}}$$

$$=\sum_{N=0}\frac{C_{2N}x^{2N}\langle g,h-g,\widetilde{1-N},\widetilde{g};q\rangle_N}{\langle 1,\widetilde{1},h,\frac{h}{2},\widetilde{\frac{h}{2}},\frac{h+1}{2},\widetilde{\frac{h+1}{2}},1\widetilde{-N}-g;q\rangle_N}=RHS.$$

∎

Theorem 2.4. *A q-analogue of* [3, p. 438 (2.8)].

(16)

$$\sum_{m,n}\frac{C_{m+n}x^{m+n}(-1)^n\langle g,h;q\rangle_m\langle g,h,\widetilde{1-\frac{m+n}{2}};q\rangle_n}{\langle 1;q\rangle_m\langle 1,\widetilde{-\frac{m+n}{2}};q\rangle_n}\mathrm{QE}\left(\frac{-n+mn}{2}\right)$$

$$=\sum_{N=0}^{\infty}C_{2N}x^{2N}\frac{\langle g,h,\widetilde{1-N},\frac{g+h}{2},\frac{g+h+1}{2},\widetilde{\frac{g+h}{2}},\widetilde{\frac{g+h+1}{2}};q\rangle_Nq^{\binom{N}{2}}}{\langle 1,\widetilde{1},g+h;q\rangle_N},\ -h-g\neq k.$$

Proof. We prove an equivalent formula.

(17)

$$\sum_{m,n}\frac{C_{m+n}x^{m+n}(-1)^n\langle g,h;q\rangle_m\langle g,h,\widetilde{1-\tfrac{m+n}{2}};q\rangle_n}{\langle 1;q\rangle_m\langle 1,\widetilde{-\tfrac{m+n}{2}};q\rangle_n}\mathrm{QE}\left(\frac{(n-3mn)}{2}+m\right)$$
$$\times\,\mathrm{QE}\left(-m^2-n^2-(m+n)(g+h)\right)=\sum_{N,n}C_Nx^N(-1)^n\langle g,h;q\rangle_{N-n}\langle g,h,\widetilde{1-\tfrac{N}{2}};q\rangle_n$$
$$\frac{\mathrm{QE}\left(\frac{(-n^2-n+Nn)}{2}+N-N^2-N(g+h)\right)}{\langle 1;q\rangle_{N-n}\langle 1,\widetilde{-\tfrac{N}{2}};q\rangle_n}=\sum_{N,n}C_Nx^N(-1)^n$$
$$\frac{\binom{N}{n}_q\langle -g+1-N,-h+1-N;q\rangle_N\langle g,h,\widetilde{1-\tfrac{N}{2}};q\rangle_n\mathrm{QE}\left(\binom{n}{2}+n(1-h-g-\tfrac{3N}{2})\right)}{\langle 1;q\rangle_N\langle -h+1-N,-g+1-N,\widetilde{-\tfrac{N}{2}};q\rangle_n}$$
$$\overset{\text{by}(12)}{=}\sum_{N=0}\frac{C_{2N}x^{2N}\langle\widetilde{1-N},\widetilde{g},1-2N-g-h;q\rangle_N\langle 1-2N-g,1-2N-h;q\rangle_{2N}\langle\tfrac{1}{2};q^2\rangle_N}{\langle g+N,\widetilde{1-N}-g,1-2N-h;q\rangle_N\langle 1;q\rangle_{2N}}$$
$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle\widetilde{1-N},\widetilde{g},g+h+N;q\rangle_N\langle g,h;q\rangle_{2N}\mathrm{QE}\left(-4N^2+2N-N(3g+2h)\right)}{\langle N+h,\widetilde{1-N}-g,g+N,1,\widetilde{1};q\rangle_N}$$
$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle g,h,\widetilde{1-N},\triangle(q;2;g+h);q\rangle_N\mathrm{QE}\left(\binom{N}{2}-4N^2+2N-N(2g+2h)\right)}{\langle 1,\widetilde{1},g+h;q\rangle_N}.$$

Finally, multiply C_n by $\mathrm{QE}\left(2\binom{n}{2}+n(g+h)\right)$. □

Theorem 2.5. *A q-analogue of* [3, p. 438 (2.9)].

(18)

$$\sum_{m,n}\frac{C_{m+n}x^{m+n}(-1)^m\langle\widetilde{1-\tfrac{m+n}{2}};q\rangle_n}{\langle 1,\nu,\sigma,\widetilde{-\tfrac{m+n}{2}};q\rangle_n\langle 1,\nu,\sigma;q\rangle_m}\mathrm{QE}\left(-\frac{n}{2}-\frac{3mn}{2}+\frac{(m+n)^2}{4}\right)$$
$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle -1+\nu+\sigma;q\rangle_{3N}\langle\widetilde{1-N};q\rangle_N(-1)^N}{\langle 1,\widetilde{1},\nu,\sigma;q\rangle_N\langle\nu,\sigma,-1+\nu+\sigma;q\rangle_{2N}},\ \nu,\sigma,\nu+\sigma-1\neq -k.$$

Proof.

(19)

$$LHS=\sum_{N=0}^{\infty}\sum_{n=0}^{N}\frac{C_Nx^N(-1)^{N-n}\langle\widetilde{1-\tfrac{N}{2}};q\rangle_n}{\langle 1,\nu,\sigma,\widetilde{-\tfrac{N}{2}};q\rangle_n\langle 1,\nu,\sigma;q\rangle_{N-n}}\mathrm{QE}\left(3\binom{n}{2}-\frac{3nN}{2}+n+\frac{N^2}{4}\right)$$

$$=\sum_{N=0}^{\infty}\frac{C_Nx^Nq^{\frac{N^2}{4}}(-1)^N}{\langle 1,\nu,\sigma;q\rangle_N}\sum_{n=0}^{N}\frac{\langle -N,-\nu+1-N,-\sigma+1-N,\widetilde{1-\tfrac{N}{2}};q\rangle_n}{\langle 1,\nu,\sigma,\widetilde{-\tfrac{N}{2}};q\rangle_n}\times$$

$$q^{n(\frac{3N}{2}-1+\nu+\sigma)}=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle\widetilde{1-N},1-\widetilde{2N-\nu},2N+\nu+\sigma-1;q\rangle_N\langle\tfrac{1}{2};q^2\rangle_N}{\langle 1,\nu,\sigma;q\rangle_{2N}\langle\sigma,\widetilde{\nu+N},1-\nu-N;q\rangle_N},$$

where we have used (12) for the q-Dixon theorem. □

Theorem 2.6. *A q-analogue of* [3, p. 440 (3.10)].

(20)

$$\sum_{m,n=0}^{\infty}\frac{C_{m+n}x^{m+n}(-1)^n\mathrm{QE}\left(\binom{m}{2}-ng\right)\langle g;q\rangle_m\langle h,\widetilde{h};q\rangle_n}{\langle 1,2g;q\rangle_m\langle 1,2h,-m-\widetilde{g+1}-n;q\rangle_n}$$

$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle h+\widetilde{g+N},\tfrac{g+h}{2},\widetilde{\tfrac{g+h}{2}},\tfrac{g+h+1}{2},\widetilde{\tfrac{g+h+1}{2}};q\rangle_N\mathrm{QE}\left(\binom{2N}{2}\right)}{\langle g+h,\widetilde{g},g+\tfrac{1}{2},\widetilde{g+\tfrac{1}{2}},h+\tfrac{1}{2},\widetilde{g+N},\widetilde{h+\tfrac{1}{2}},\widetilde{1},1;q\rangle_N}.$$

Proof. We prove the equivalent formula

$$\sum_{m,n=0}^{\infty}\frac{C_{m+n}x^{m+n}(-1)^n\mathrm{QE}\left(-\binom{n}{2}-mn+mg\right)\langle g;q\rangle_m\langle h,\widetilde{h};q\rangle_n}{\langle 1,2g;q\rangle_m\langle 1,2h,-m-\widetilde{g+1}-n;q\rangle_n}$$

(21)

$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle h+\widetilde{g+N},\tfrac{g+h}{2},\widetilde{\tfrac{g+h}{2}},\tfrac{g+h+1}{2},\widetilde{\tfrac{g+h+1}{2}};q\rangle_Nq^{2gN}}{\langle g+h,\widetilde{g},g+\tfrac{1}{2},\widetilde{g+\tfrac{1}{2}},h+\tfrac{1}{2},\widetilde{g+N},\widetilde{h+\tfrac{1}{2}},\widetilde{1},1;q\rangle_N}.$$

(22)

$$LHS=\sum_{N=0}^{\infty}\sum_{n=0}^{N}\frac{C_N\langle g;q\rangle_{N-n}\langle h,\widetilde{h};q\rangle_n x^N(-1)^n\mathrm{QE}\left(-\binom{n}{2}-(N-n)n+(N-n)g\right)}{\langle 1,2g;q\rangle_{N-n}\langle 1,2h,\widetilde{-g+1}-N;q\rangle_n}=$$

$$\sum_{N=0}^{\infty}\sum_{n=0}^{N}\frac{C_N\langle -g+1-N;q\rangle_N\langle h,\widetilde{h},-2g+1-N;q\rangle_n x^N(-1)^n\mathrm{QE}\left(\binom{n}{2}+n(1-N)\right)}{\langle 1;q\rangle_{N-n}\langle -2g+1-N;q\rangle_N\langle -g+1-N,1,2h,\widetilde{-g+1}-N;q\rangle_n}=$$

$$\sum_{N=0}\frac{C_N\langle -g+1-N;q\rangle_N x^N}{\langle 1,-2g+1-N;q\rangle_N}\sum_{n=0}^{N}\frac{\binom{N}{n}_q\langle h,\widetilde{h},-2g+1-N;q\rangle_n(-1)^n\mathrm{QE}\left(\binom{n}{2}+n(1-N)\right)}{\langle -g+1-N,2h,\widetilde{-g+1}-N;q\rangle_n}$$

$$\overset{\text{by}(13)}{=}\sum_{N=0}^{\infty}\frac{C_{2N}\langle -g+1-2N;q\rangle_{2N}x^{2N}\langle\frac{1}{2},h+g+N;q^2\rangle_N}{\langle 1,-2g+1-2N;q\rangle_{2N}\langle\frac{1}{2}+h,g+N;q^2\rangle_N}=$$

$$\sum_{N=0}^{\infty}\frac{C_{2N}\langle g;q\rangle_{2N}x^{2N}\langle\frac{1}{2},h+g+N;q^2\rangle_N q^{2gN}}{\langle 1,2g;q\rangle_{2N}\langle\frac{1}{2}+h,g+N;q^2\rangle_N}$$

$$=\sum_{N=0}^{\infty}\frac{C_{2N}x^{2N}\langle \widetilde{h+g+N},\frac{g+h}{2},\widetilde{\frac{g+h}{2}},\frac{g+h+1}{2},\widetilde{\frac{g+h+1}{2}};q\rangle_N q^{2gN}}{\langle g+h,\widetilde{g},g+\frac{1}{2},\widetilde{g+\frac{1}{2}},h+\frac{1}{2},\widetilde{g+N},\widetilde{h+\frac{1}{2}},\widetilde{1},1;q\rangle_N}.$$

Finally, multiply C_n by $\mathrm{QE}\left(\binom{n}{2}-ng\right)$. □

2.4. **Reduction formulas.**

Theorem 2.7. *A q-analogue of a reduction formula for the second Appell function.*

$$\sum_{m,n=0}^{\infty}\frac{\langle\lambda;q\rangle_{m+n}x^{m+n}(-1)^n\langle g;q\rangle_m\langle g,\widetilde{1-\frac{m+n}{2}};q\rangle_n\mathrm{QE}\left(\frac{-mn-n}{2}\right)}{\langle 1,h;q\rangle_m\langle 1,h,\widetilde{-\frac{m+n}{2}};q\rangle_n}$$
(23)
$$=\ {}_7\phi_7\left[\begin{array}{c}\triangle(q;2;\lambda),g,h-g\\ \triangle(q;2;h),h,\widetilde{1},\infty\end{array}|q;-x^2q^g||\begin{array}{c}\langle\widetilde{1-k};q\rangle_k\\ -\end{array}\right].$$

Proof. Put $C_k=\langle\lambda;q\rangle_k$ in (14). □

Remark 2. The righthand side of formulas (23) and (25) converge quicker than the LHS because of the q-power with negative exponent, the double sum and the minus sign. The other formulas in this section have similar properties.

Remark 3. We can't use the q-analogue of Burchnall-Chaundy [2, (43)] from [5] (or [9]) to produce another q-analogue of a reduction formula for the second Appell function, since these formulas use NWA. The proof would have worked if the formula used JHC, like in [13, p. 77 (50)]; this is however not possible as shown in the appendix.

Theorem 2.8. *A q-analogue of a reduction formula for the third Appell function. By using vectors, this formula can easily be extended to a q-analogue of* [26, p. 31 (48)].

$$\sum_{m,n=0}^{\infty}\frac{x^{m+n}(-1)^n\langle g,h;q\rangle_m\langle g,h,\widetilde{1-\tfrac{m+n}{2}};q\rangle_n}{\langle\mu;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1,\widetilde{-\tfrac{m+n}{2}};q\rangle_n}\mathrm{QE}\left(\frac{-n+mn}{2}\right)$$
$$= {}_7\phi_7\left[\begin{matrix} g,h,\triangle(q;2;g+h) \\ \triangle(q;2;\mu),g+h,\widetilde{1},\infty\end{matrix}\,|q;-x^2||\,\begin{matrix}\langle\widetilde{1-k};q\rangle_k \\ -\end{matrix}\right]. \tag{24}$$

Proof. Put $C_k=\frac{1}{\langle\mu;q\rangle_k}$ in (16). □

Corollary 2.9. *A q-analogue of* [3, p. 439 (3.4)] *and* [26, p. 31 (46)]

$$\Phi^{p:2;3}_{p:1;2}\left[\begin{matrix}\vec{\lambda}:g,\infty;g,\infty \\ \vec{\mu}:h;h\end{matrix}\,|q;x,-xq^{-\frac{1}{2}}||\,\begin{matrix}\langle\widetilde{1-\tfrac{m+n}{2}};q\rangle_n q^{-\frac{mn}{2}} \\ \langle\widetilde{-\tfrac{m+n}{2}};q\rangle_n\end{matrix}\right]=$$
$${}_{6+4p}\phi_{6+4p}\left[\begin{matrix}\triangle(q;2;\vec{\lambda}),g,h-g,3\infty \\ \triangle(q;2;\vec{\mu},h),h,\widetilde{1}\end{matrix}\,|q;-x^2q^g||\,\begin{matrix}\langle\widetilde{1-k};q\rangle_k \\ -\end{matrix}\right]. \tag{25}$$

Proof. Put $C_n=\frac{\langle\vec{\lambda};q\rangle_n}{\langle\vec{\mu};q\rangle_n}$ in (14). □

Corollary 2.10. *A q-analogue of* [3, p. 439 (3.5)]

$$\Phi^{p:1;2}_{p:1;2}\left[\begin{matrix}\vec{\lambda}:g;h,\widetilde{h} \\ \vec{\mu}:2g;2h\end{matrix}\,|q;-x,x||\,\begin{matrix}q^{mn} \\ \langle\widetilde{m+g};q\rangle_n\end{matrix}\right]=$$
$${}_{5+4p}\phi_{8+4p}\left[\begin{matrix}\triangle(q;2;\vec{\lambda},g+h) \\ \triangle(q;2;\vec{\mu}),g+h,g+\tfrac{1}{2},h+\tfrac{1}{2},\widetilde{g+\tfrac{1}{2}},\widetilde{h+\tfrac{1}{2}},\widetilde{1},\widetilde{g}\end{matrix}\right.$$
$$\left.|q;x^2q||\,\begin{matrix}\langle h+\widetilde{g+k};q\rangle_k \\ \langle\widetilde{k+g};q\rangle_k\end{matrix}\right]. \tag{26}$$

Proof. Put $C_n=\frac{\langle\vec{\lambda};q\rangle_n}{\langle\vec{\mu};q\rangle_n}$ in (20). □

Theorem 2.11. *A q-analogue of* [3, p. 439 (3.8)] *and* [26, p. 32 (50)]

(27)

$$\sum_{m,n}\frac{\langle\vec{\lambda};q\rangle_{m+n}x^{m+n}(-1)^m\langle\widetilde{1-\tfrac{m+n}{2}};q\rangle_n}{\langle\vec{\mu};q\rangle_{m+n}\langle 1,\nu,\sigma,\widetilde{-\tfrac{m+n}{2}};q\rangle_n\langle 1,\nu,\sigma;q\rangle_m}\mathrm{QE}\left(-\frac{n}{2}-\frac{3mn}{2}+\frac{(m+n)^2}{4}\right)=$$
$${}_{16+4p}\phi_{15+4p}\left[\begin{matrix}\triangle(q;2;\vec{\lambda}),\triangle(q;3;\nu+\sigma-1),9\infty \\ \triangle(q;2;\vec{\mu},\nu,\sigma,\nu+\sigma-1),\nu,\sigma,\widetilde{1}\end{matrix}\,|q;-x^2||\,\begin{matrix}\langle\widetilde{1-k};q\rangle_k \\ -\end{matrix}\right].$$

Proof. Put $C_n=\frac{\langle\vec{\lambda};q\rangle_n}{\langle\vec{\mu};q\rangle_n}$ in (18).

□

This last formula is the crown of our efforts in this section, and beautifully unites the notation used so far. The formula [26, p. 32 (50)] is also the last one in the corresponding chapter. We will come back to more q-analogues from [26] in later papers.

3. Some further q-analogues of Srivastavas formulas

In this section we show more examples of our method and also extend some of the notions of q-hypergeometric series to multiple q-series. Although the title of the papers [29] and [30] is *a class of finite q-series*, their main content is concerned with the hypergeometric case.

Theorem 3.1. *A q-analogue of* [29, p. 17, 2.1].

(28)
$$\sum_{n=0}^{N}\frac{(-1)^n q^{\binom{n}{2}}}{\langle 1;q\rangle_n\langle 1;q\rangle_{N-n}}\Phi_{p:r;u}^{p:r+1;u+1}\left[\begin{array}{c}(a):-n,(c);-N+n,(\alpha)\\(b):(d);(\beta)\end{array}|q;xq,yq^{-n}\right]=$$
$$\frac{x^N\langle (a),(c);q\rangle_N}{\langle (b),(d),1;q\rangle_N}\,{}_{u+r+1}\phi_{u+r}\left[\begin{array}{c}(1-d-N),-N,(\alpha)\\(1-c-N),(\beta)\end{array}|q;q^{D-C}\frac{y}{x}\right],$$

where $D=\sum_j d_j,\ C=\sum_j c_j$.

Proof.

(29)

$$\sum_{n=0}^{N}\frac{(-1)^n q^{\binom{n}{2}}}{\langle 1;q\rangle_n\langle 1;q\rangle_{N-n}}\Phi_{p:r;u}^{p:r+1;u+1}\left[\begin{array}{c}(a):-n,(c);-N+n,(\alpha)\\(b):(d);(\beta)\end{array}|q;xq,yq^{-n}\right]=$$

$$\sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{m=0}^{N-n}\frac{(-1)^{l+m+n}\langle(a);q\rangle_{l+m}\langle(c);q\rangle_l\langle(\alpha);q\rangle_m x^l y^m}{\langle(b);q\rangle_{l+m}\langle(d);q\rangle_l\langle(\beta);q\rangle_m\langle 1;q\rangle_l\langle 1;q\rangle_m\langle 1;q\rangle_{n-l}\langle 1;q\rangle_{N-n-m}}$$

$$\times\,\mathrm{QE}\left(\binom{l}{2}+\binom{m}{2}+\binom{n}{2}+l-nl-mN\right)=$$

$$\sum_{l,m=0}^{\infty}\frac{(-1)^{l+m}\langle(a);q\rangle_{l+m}\langle(c);q\rangle_l\langle(\alpha);q\rangle_m x^l y^m\mathrm{QE}\left(\binom{m}{2}-mN\right)}{\langle(b);q\rangle_{l+m}\langle(d);q\rangle_l\langle(\beta);q\rangle_m\langle 1;q\rangle_l\langle 1;q\rangle_m}$$

$$\sum_{n=l}^{N-m}\frac{(-1)^n\mathrm{QE}\left(\binom{n-l}{2}\right)}{\langle 1;q\rangle_{n-l}\langle 1;q\rangle_{N-n-m}}$$

$$=\sum_{l,m=0}^{l+m\leq N}\frac{(-1)^m\langle(a);q\rangle_{l+m}\langle(c);q\rangle_l\langle(\alpha);q\rangle_m x^l y^m\mathrm{QE}\left(\binom{m}{2}-mN\right)}{\langle(b);q\rangle_{l+m}\langle(d);q\rangle_l\langle(\beta);q\rangle_m\langle 1;q\rangle_l\langle 1;q\rangle_m\langle 1;q\rangle_{N-l-m}}$$

$$\sum_{n=0}^{N-l-m}(-1)^n\mathrm{QE}\left(\binom{n}{2}\right)\binom{N-l-m}{n}_q$$

$$=\sum_{l+m=N}\frac{(-1)^m\langle(a);q\rangle_{l+m}\langle(c);q\rangle_l\langle(\alpha);q\rangle_m x^l y^m\mathrm{QE}\left(\binom{m}{2}-mN\right)}{\langle(b);q\rangle_{l+m}\langle(d);q\rangle_l\langle(\beta);q\rangle_m\langle 1;q\rangle_l\langle 1;q\rangle_m\langle 1;q\rangle_{N-l-m}}$$

$$=\frac{\langle(a);q\rangle_N x^N}{\langle(b);q\rangle_N}\sum_{m=0}^{N}\frac{(-y)^m\langle(c);q\rangle_{N-m}\langle(\alpha);q\rangle_m\mathrm{QE}\left(\binom{m}{2}-mN\right)}{x^m\langle(d);q\rangle_{N-m}\langle(\beta);q\rangle_m\langle 1;q\rangle_m\langle 1;q\rangle_{N-m}}$$

$$=\frac{\langle(a),(c);q\rangle_N x^N}{\langle(b),(d),1;q\rangle_N}\sum_{m=0}^{N}\frac{y^m\langle -N,(\alpha),(1-d-N);q\rangle_m\mathrm{QE}\left(m(\sum d_j-\sum c_j)\right)}{x^m\langle(\beta),(1-c-N),1;q\rangle_m}\;.$$

□

Theorem 3.2. *A q-analogue of* [30, p. 41, 3.1].

(30)
$$\sum_{n=0}^{N}\frac{\langle\lambda;q\rangle_n\langle\mu;q\rangle_{N-n}}{\langle 1;q\rangle_n\langle 1;q\rangle_{N-n}}\sum_{l=0}^{\infty}\sum_{m=0}^{N-n}A_{l+m}B_lC_mq^{n\mu}\frac{x^ly^m\langle -N+n;q\rangle_m}{\langle\mu,1;q\rangle_m\langle 1;q\rangle_l}=$$
$$\frac{\langle\lambda+\mu;q\rangle_N}{\langle 1;q\rangle_N}\sum_{l=0}^{\infty}\sum_{m=0}^{N}A_{l+m}B_lC_m\frac{x^ly^m\langle -N;q\rangle_m}{\langle\lambda+\mu,1;q\rangle_m\langle 1;q\rangle_l},$$

where A, B, C are arbitrary sequences of complex numbers.

Proof.

$$LHS = \sum_{n=0}^{N} \frac{\langle \lambda; q\rangle_n \langle \mu; q\rangle_{N-n}}{\langle 1; q\rangle_n} \sum_{l=0}^{\infty} \sum_{m=0}^{N-n} A_{l+m} B_l C_m$$

$$\frac{x^l(-y)^m \mathrm{QE}\left(\binom{m}{2} + m(n-N) + n\mu\right)}{\langle \mu, 1; q\rangle_m \langle 1; q\rangle_l \langle 1; q\rangle_{N-m-n}} =$$

$$\sum_{l,m=0}^{\infty} A_{l+m} B_l C_m \frac{x^l(-y)^m}{\langle 1; q\rangle_l \langle 1; q\rangle_m} \sum_{n=0}^{N-m} \mathrm{QE}\left(\binom{m}{2} - mN\right) \tag{31}$$

$$\frac{\langle \lambda; q\rangle_n \langle \mu + m; q\rangle_{N-m-n}}{\langle 1; q\rangle_n \langle 1; q\rangle_{N-m-n}} q^{n\mu} = \sum_{l,m=0}^{\infty} A_{l+m} B_l C_m \frac{x^l(-y)^m}{\langle 1; q\rangle_l \langle 1; q\rangle_m}$$

$$\mathrm{QE}\left(\binom{m}{2} - mN\right) \frac{\langle \lambda + \mu + m; q\rangle_{N-m}}{\langle 1; q\rangle_{N-m}} = RHS.$$

□

3.1. **Balanced and very-well-poised double q-series.** We now come to a definition which is new even for the case $q = 1$. We illustrate with two q-analogues of [28]. We remind that this concept has occurred before, in [1, p.456 (7)] the following summation formula was proved:

$$\Phi^{1:2}_{1:1}\left[\begin{matrix} \alpha : -m, \beta; -n, \beta' \\ \beta + \beta' : 1 + \alpha - \beta' - m; 1 + \alpha - \beta - n \end{matrix} |q; q, q\right] = \frac{\langle \beta + \beta' - \alpha; q\rangle_{m+n} \langle \beta'; q\rangle_m \langle \beta; q\rangle_n}{\langle \beta + \beta'; q\rangle_{m+n} \langle \beta' - \alpha; q\rangle_m \langle \beta - \alpha; q\rangle_n}. \tag{32}$$

Theorem 3.3. *We use the following notation:*

$$\begin{aligned} &(\alpha) \equiv (a, b, c, d, 1 + \tfrac{1}{2}a, \widetilde{1 + \tfrac{1}{2}a}, e), \\ &(\beta) \equiv (1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, \tfrac{1}{2}a, \widetilde{\tfrac{1}{2}a}). \end{aligned} \tag{33}$$

Assume that

$$2a + M + N + 1 \equiv b + c + d + e \mod \frac{2\pi i}{\log q}. \tag{34}$$

Then we have a q-analogue of [28, p. 245 (1.4)]*:*

$$\Phi^{7:1;1}_{7:0;0}\left[\begin{matrix} (\alpha) : -M; -N \\ (\beta), 1 + a + M + N : -; - \end{matrix} |q; q^{1-N}, q\right] = \frac{\langle 1 + a, 1 + a - b - c, 1 + a - b - d, 1 + a - c - d; q\rangle_{M+N}}{\langle 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - b - c - d; q\rangle_{M+N}}. \tag{35}$$

Proof. The following theorem is known from [5, p. 219 (74)]:

$$\sum_{m,n=0}^{\infty} \frac{C_{m+n}x^{m+n}\langle \nu;q\rangle_m \langle \sigma;q\rangle_n q^{-n\sigma}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} = \sum_{n=0}^{\infty} \frac{C_n x^n \langle \nu+\sigma;q\rangle_n}{\langle 1;q\rangle_n} q^{-n\sigma}. \tag{36}$$

Put $C_n = \frac{\langle(\alpha);q\rangle_n}{\langle(\beta);q\rangle_n}$, $x = q^{1-N}$, $\nu = -M$ and $\sigma = -N$.

Assume that

$$2a+n+1 \equiv b+c+d+e \bmod \frac{2\pi i}{\log q}. \tag{37}$$

Then

$$\begin{aligned} &{}_8\phi_7\left[\begin{array}{c} a,b,c,d,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},e,-n \\ (\beta),1+a+n \end{array}|q;q\right] = \\ &= \frac{\langle 1+a,1+a-b-c,1+a-b-d,1+a-c-d;q\rangle_n}{\langle 1+a-b,1+a-c,1+a-d,1+a-b-c-d;q\rangle_n}, \end{aligned} \tag{38}$$

where $n = 0,1,2,\ldots$.

Thus we get

$$\begin{aligned} &\Phi^{7:1;1}_{7:0;0}\left[\begin{array}{c} (\alpha):-M;-N \\ (\beta),1+a+M+N:-;- \end{array}|q;q^{1-N},q\right] = \\ &{}_8\phi_7\left[\begin{array}{c} a,b,c,d,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},e,-M-N \\ (\beta),1+a+M+N \end{array}|q;q\right] = \\ &\frac{\langle 1+a,1+a-b-c,1+a-b-d,1+a-c-d;q\rangle_{M+N}}{\langle 1+a-b,1+a-c,1+a-d,1+a-b-c-d;q\rangle_{M+N}}. \end{aligned} \tag{39}$$

□

The series (35) is called balanced with respect to both summation indices. The series (35) and (42) are called very-well-poised if we sum the two m and n factorials together with the last factorial in the denominator.

We will need the following lemma:

Lemma 3.4.

$$\begin{aligned} &{}_6\phi_4\left[\begin{array}{c} a,b,c,\infty,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a} \\ 1+a-b,1+a-c,\frac{1}{2}a,\widetilde{\frac{1}{2}a} \end{array}|q;q^{-b-c}\right] = \\ &= \Gamma_q\left[\begin{array}{c} 1+a-b,1+a-c \\ 1+a,1+a-b-c \end{array}\right]. \end{aligned} \tag{40}$$

Proof. Let $d \to \infty$ in the equation

$$
{}_6\phi_5 \left[\begin{array}{c} a, b, c, d, 1+\frac{1}{2}a, \widetilde{1+\frac{1}{2}a} \\ 1+a-b, 1+a-c, 1+a-d, \frac{1}{2}a, \widetilde{\frac{1}{2}a} \end{array} | q; q^{1+a-b-c-d} \right] = \\
= \Gamma_q \left[\begin{array}{c} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ 1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{array} \right]. \tag{41}
$$

□

Theorem 3.5. *A q-analogue of* [28, p. 247 (2.8)]*:*

$$
\Phi^{5:1;1}_{4:0;0} \left[\begin{array}{c} \alpha, 1+\frac{\alpha}{2}, \widetilde{1+\frac{\alpha}{2}}, \beta, \infty : \nu; \sigma \\ \frac{\alpha}{2}, \widetilde{\frac{\alpha}{2}}, 1+\alpha-\beta, 1+\alpha-\nu-\sigma : -; -; \end{array} | q; q^{-\beta-\nu}, q^{-\sigma-\beta-\nu} \right] = \\
\Gamma_q \left[\begin{array}{c} 1+\alpha-\beta, 1+\alpha-\nu-\sigma \\ 1+\alpha, 1+\alpha-\beta-\nu-\sigma \end{array} \right]. \tag{42}
$$

Proof. Put $C_n \equiv \frac{\langle \alpha, 1+\frac{\alpha}{2}, \widetilde{1+\frac{\alpha}{2}}, \beta, \infty; q\rangle_n}{\langle \frac{\alpha}{2}, \widetilde{\frac{\alpha}{2}}, 1+\alpha-\beta, 1+\alpha-\nu-\sigma; q\rangle_n}$, $x \equiv q^{-\beta-\nu}$ in (36) and use formula (40). □

We conclude this section by two q-analogues of [24].

Theorem 3.6. *A q-analogue of* [24, p. 242 (4)].

$$
\Phi^{3:1;1}_{3:0;0} \left[\begin{array}{c} \frac{c}{2}, \widetilde{\frac{c}{2}}, -n : \nu; \sigma; \\ \frac{-n+1+\nu+\sigma}{2}, \widetilde{\frac{-n+1+\nu+\sigma}{2}}, c : -; -; \end{array} | q; q^{\sigma+1}, q \right] = \\
q^{\frac{n(\nu+\sigma)}{2}} \Gamma_{q^2} \left[\begin{array}{c} \frac{1}{2}, \frac{c+1}{2}, \frac{1+\nu+\sigma-n}{2}, \frac{1-(\nu+\sigma)+n+c}{2} \\ \frac{\nu+\sigma+1}{2}, \frac{-n+1}{2}, \frac{1-(\nu+\sigma)+c}{2}, \frac{1+n+c}{2} \end{array} \right]. \tag{43}
$$

Proof. Use formula (36) together with the following equation from [8, p.172]:

$$
{}_4\phi_3 \left[\begin{array}{c} \frac{c}{2}, \widetilde{\frac{c}{2}}, a, -n \\ \frac{-n+1+a}{2}, \widetilde{\frac{-n+1+a}{2}}, c \end{array} | q; q \right] = q^{\frac{na}{2}} \Gamma_{q^2} \left[\begin{array}{c} \frac{1}{2}, \frac{c+1}{2}, \frac{1+a-n}{2}, \frac{1-a+n+c}{2} \\ \frac{a+1}{2}, \frac{-n+1}{2}, \frac{1-a+c}{2}, \frac{1+n+c}{2} \end{array} \right]. \tag{44}
$$

□

We need a generalization of (36) for the proof of the following theorem.

Lemma 3.7. [6] *A q-analogue of Panda* [16] *and Singhal* [21]. *If* $\{C_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ *are sequences of arbitrary complex numbers, then*

$$\begin{aligned}&\sum_{\vec{m}} \frac{C_{m_1+\ldots+m_n}\prod_{j=1}^n x^{m_j}\langle\alpha_j;q\rangle_{m_j}}{\prod_{j=1}^n\langle 1;q\rangle_{m_j}}\mathrm{QE}\left(-\sum_{k=1}^n m_k\sum_{l=2}^k\alpha_l\right)= \\ &\sum_{N=0}^{\infty}\frac{C_N x^N\langle\sum_{k=1}^n\alpha_k;q\rangle_N}{\langle 1;q\rangle_N}\mathrm{QE}\left(-N\sum_{l=2}^n\alpha_l\right).\end{aligned} \tag{45}$$

Theorem 3.8. *A q-analogue of* [24, p. 244 (12)].

(46)

$$\Phi^{3:1;\ldots;1}_{3:0;\ldots;0}\left[\begin{matrix}\frac{c}{2},\widetilde{\frac{c}{2}},\widetilde{-j}:\gamma_1;\ldots;\gamma_n\\ \frac{-j+1+\gamma}{2},\frac{-j+1+\gamma}{2},c:-;\ldots;-\end{matrix}\;\Big|q||\mathrm{QE}\left(\sum_{k=1}^n m_k\left(1+\gamma-\sum_{l=1}^k\gamma_l\right)\right)\right]$$

$$=q^{\frac{j\gamma}{2}}\;\Gamma_{q^2}\left[\begin{matrix}\frac{1}{2},\frac{c+1}{2},\frac{1+\gamma-j}{2},\frac{1-\gamma+j+c}{2}\\ \frac{\gamma+1}{2},\frac{-j+1}{2},\frac{1-\gamma+c}{2},\frac{1+j+c}{2}\end{matrix}\right],\;\gamma=\sum_{k=1}^n\gamma_k.$$

Proof. Use formula (45) together with (44). ☐

The notion balanced q-hypergeometric series of three variables was introduced in [20]. We will come back to this in another article.

4. Certain q-summation formulas

Srivastava [22] and Panda & Srivastava [17] have systematically collected and generalized a number of related summation formulas known from the literature. Our task in this section is to find symmetric q-analogues of these formulas, which always occur in pairs. In certain exceptional cases the convergence in the formulas is not so good, we then replace the equality sign by the sign for formal equality, $\cong$.

We assume throughout that $M = km + ln$ and $\{C(m,n)\}_{m,n=0}^{\infty}$ is a sequence of bounded complex numbers. Then

Theorem 4.1. *Compare* [17, (4) p. 244] *and* [22, (9) p. 28].

(47)

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta+r;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}q^{\binom{r}{2}-rM}=$$

$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{-NM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.$$

Proof.

(48)

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta+r;q\rangle_M\frac{x^m y^n q^{\binom{r}{2}-rM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$

$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{r(-\alpha+\beta-M+N)}=$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$

$$q^{r(-\alpha+\beta-M+N)}=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-\alpha;q\rangle_N}=$$

$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{-NM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.$$

□

Theorem 4.2. *Compare* [17, (4) p. 244].

(49)

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta+r;q\rangle_M\frac{x^m y^n q^{\binom{r}{2}+r(\alpha-\beta-N+1)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}=$$

$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.$$

Proof.

(50)

$$LHS = \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} C(m,n)\langle 1-\beta+r;q\rangle_M \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n}$$

$$\mathrm{QE}\left(\binom{r}{2} + r(\alpha-\beta-N+1)\right) = \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}$$

$$\sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_r} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r} q^r =$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\langle 1-\beta;q\rangle_M \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha;q\rangle_r} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r} q^r =$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n \langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-\alpha;q\rangle_N} q^{N(1-\beta+M)} =$$

$$\frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n \langle 1-\beta;q\rangle_M q^{N(1-\beta)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.$$

□

Theorem 4.3. *Compare* [17, (5) p. 244] *and* [22, (8) p. 28].

(51)

$$\sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M} \frac{x^m y^n q^{\binom{r}{2}}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} =$$

$$\frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n \langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{1}{\langle 1-\alpha+N;q\rangle_M}.$$

Proof.

$$LHS = \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M}\frac{x^m y^n q^{\binom{r}{2}}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \tag{52}$$

$$= \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_{r+M}} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r} q^{r(-\alpha+\beta+N)} =$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{\langle 1-\beta;q\rangle_M}{\langle 1-\alpha;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$

$$q^{r(-\alpha+\beta+N)} = \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n \langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m \langle 1;q\rangle_n}\frac{\langle -\alpha+\beta;q\rangle_N}{\langle 1-\alpha;q\rangle_M \langle 1-\alpha+M;q\rangle_N} =$$

$$\frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n \langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m \langle 1;q\rangle_n}\frac{1}{\langle 1-\alpha+N;q\rangle_M}.$$

□

Theorem 4.4. *Compare* [17, (5) p. 244]*.*

$$\sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \tag{53}$$

$$\mathrm{QE}\left(\binom{r}{2} + r(\alpha-\beta-N+1)\right) =$$

$$\frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n \langle 1-\beta;q\rangle_M q^{N(1-\beta+M)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1-\alpha+N;q\rangle_M}.$$

Proof.

(54)

$$LHS = \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n}$$

$$\mathrm{QE}\left(\binom{r}{2}+r(\alpha-\beta-N+1)\right) = \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n}$$

$$\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_{r+M}}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r = \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty} C(m,n)\frac{\langle 1-\beta;q\rangle_M}{\langle 1-\alpha;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n}$$

$$\sum_{r=0}^{N}\frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r = \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m \langle 1;q\rangle_n}$$

$$\frac{\langle -\alpha+\beta;q\rangle_N q^{N(1-\beta+M)}}{\langle 1-\alpha;q\rangle_M \langle 1+\alpha+M;q\rangle_N} =$$

$$\frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)}\sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n \langle 1-\beta;q\rangle_M q^{N(1-\beta+M)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1-\alpha+N;q\rangle_M}.$$

□

Theorem 4.5. *Compare* [17, (6) p. 244] *and* [22, (10) p. 29].

(55)

$$\sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty} C(m,n)\frac{\langle \alpha-r;q\rangle_M}{\langle \beta-r;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n}\mathrm{QE}\left(\binom{r}{2}\right)$$

$$= \frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)}\sum_{m,n=0}^{\infty} C(m,n)\frac{x^m y^n q^{NM}}{\langle 1;q\rangle_m \langle 1;q\rangle_n}\frac{\langle \alpha-N;q\rangle_M}{\langle \beta;q\rangle_M}.$$

Proof.

(56)
$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle\alpha-r;q\rangle_M}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}}}{\langle1;q\rangle_m\langle1;q\rangle_n}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle1;q\rangle_m\langle1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\alpha;q\rangle_{M-r}}{\langle\beta;q\rangle_{M-r}}\frac{\langle-N;q\rangle_r}{\langle1;q\rangle_r}q^{rN}=$$
$$\frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle1;q\rangle_m\langle1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle1-\beta-M;q\rangle_r}{\langle1-\alpha-M;q\rangle_r}\frac{\langle-N;q\rangle_r}{\langle1;q\rangle_r}$$
$$q^{r(-\alpha+\beta+N)}=\frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle1;q\rangle_m\langle1;q\rangle_n}\frac{\langle-\alpha+\beta;q\rangle_N}{\langle1-\alpha-M;q\rangle_N}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^nq^{NM}}{\langle1;q\rangle_m\langle1;q\rangle_n}\frac{\langle\alpha-N;q\rangle_M}{\langle\beta;q\rangle_M}.$$

□

Theorem 4.6. *Compare* [17, (6) p. 244].

(57)
$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle\alpha-r;q\rangle_M}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}+r(\alpha-\beta-N+1)}}{\langle1;q\rangle_m\langle1;q\rangle_n}$$
$$=\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^nq^{N(1-\beta)}}{\langle1;q\rangle_m\langle1;q\rangle_n}\frac{\langle\alpha-N;q\rangle_M}{\langle\beta;q\rangle_M}.$$

Proof.

(58)

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle\alpha-r;q\rangle_M}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}+r(\alpha-\beta-N+1)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$

$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\alpha;q\rangle_{M-r}}{\langle\beta;q\rangle_{M-r}}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{r(\alpha-\beta+1)}=$$

$$\frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta-M;q\rangle_r}{\langle 1-\alpha-M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r$$

$$=\frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta;q\rangle_N q^{N(1-\beta-M)}}{\langle 1-\alpha-M;q\rangle_N}=$$

$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle\alpha-N;q\rangle_M q^{N(1-\beta)}}{\langle\beta;q\rangle_M}.$$

□

Theorem 4.7. *Compare* [17, (7) p. 244] *and* [22, (11) p. 29].

(59)
$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle\beta-r;q\rangle_M}\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}q^{\binom{r}{2}}\cong$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^nq^{\binom{r}{2}}}{\langle\beta;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle -\alpha+\beta;q\rangle_M}.$$

Proof.

(60)

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$

$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle -N;q\rangle_r}{\langle\alpha-r;q\rangle_r\langle 1;q\rangle_r}\frac{q^{rN}}{\langle\beta;q\rangle_{M-r}}=$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)\langle\beta;q\rangle_M}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta-M;q\rangle_r}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$

$$q^{r(-\alpha+\beta+M+N)}=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle\beta;q\rangle_M}\frac{\langle -\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha;q\rangle_N}=$$

$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n}{\langle\beta;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle -\alpha+\beta;q\rangle_M}.$$

□

Remark 4. The convergence in formula (59) is not so good. However, this formula works well in a number of special cases. One example is

(61)

$$C(m,n)=1,q=.85,\alpha=5.3,\beta=5.543,N=4,x=.2,y=.176.$$

In this case the difference LHS-RHS in (59) is 4.09273×10^{-12} for $m=n=60$.

Theorem 4.8. *Compare* [17, (7) p. 244].

(62)

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle\beta-r;q\rangle_M}\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}q^{\binom{r}{2}+r(1+\alpha-\beta-M-N)}=$$

$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^nq^{N(1-\beta-M)}}{\langle\beta;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle -\alpha+\beta;q\rangle_M}.$$

Proof.

(63)

$$LHS = \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} \frac{C(m,n)}{\langle\beta-r;q\rangle_M} \frac{x^m y^n q^{r(1+\alpha-\beta-M-N}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} q^{\binom{r}{2}+1-2M}$$

$$= \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle -N;q\rangle_r}{\langle \alpha-r;q\rangle_r \langle 1;q\rangle_r} \frac{q^{r(1+\alpha-\beta-M)}}{\langle\beta;q\rangle_{M-r}} =$$

$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)\langle\beta;q\rangle_M} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta-M;q\rangle_r}{\langle 1-\alpha;q\rangle_r} \frac{\langle -N;q\rangle_r q^r}{\langle 1;q\rangle_r}$$

$$= \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n) x^m y^n q^{N(1-\beta-M)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle\beta;q\rangle_M} \frac{\langle -\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha;q\rangle_N} =$$

$$\frac{\langle\beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n) x^m y^n q^{N(1-\beta-M)}}{\langle\beta;q\rangle_M \langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle -\alpha+\beta;q\rangle_M}.$$

□

Theorem 4.9. *Compare* [17, (8) p. 244] *and* [22, (12) p. 29].

(64)

$$\sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} C(m,n) \langle\alpha-r;q\rangle_M \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \mathrm{QE}\left(\binom{r}{2}\right) =$$

$$\frac{\langle\beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n \langle\alpha-N;q\rangle_M}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.$$

Proof.

(65)

$$
\begin{aligned}
&LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha-r;q\rangle_M\frac{x^m y^n q^{\binom{r}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\beta-r;q\rangle_r\langle\alpha;q\rangle_{M-r}\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{rN}=\\
&\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_r q^{r(-\alpha+\beta-M+N)}}{\langle 1-\alpha-M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}\\
&=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^m y^n\langle\alpha;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-M-\alpha;q\rangle_N}=\\
&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^m y^n\langle\alpha-N;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.
\end{aligned}
$$

□

Theorem 4.10. *Compare* [17, (8) p. 244].

(66)

$$
\begin{aligned}
&\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha-r;q\rangle_M\frac{x^m y^n q^{\binom{r}{2}+r(1+\alpha-\beta+M-N)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\cong\\
&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.
\end{aligned}
$$

Proof.

(67)
$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha-r;q\rangle_M\frac{x^my^nq^{\binom{r}{2}+r(1+\alpha-\beta+M-N)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\beta-r;q\rangle_r\langle\alpha;q\rangle_{M-r}\langle -N;q\rangle_rq^{r(1+\alpha-\beta+M)}}{\langle 1;q\rangle_r}=$$
$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha;q\rangle_M\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_rq^r}{\langle 1-\alpha-M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n\langle\alpha;q\rangle_Mq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-M-\alpha;q\rangle_N}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{x^my^n\langle\alpha-N;q\rangle_Mq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.$$

□

Remark 5. The convergence in formula (66) is not so good. However, this formula works somehow in a number of special cases. One example is

(68)
$$C(m,n)=1,q=.99,\alpha=4.3,\beta=5.543,N=4,x=.1,y=.076.$$

In this case the difference LHS-RHS in (66) is 0.0000127104 for $m=n=20$.

Theorem 4.11. *Compare* [17, (9) p. 244] *and* [22, (13) p. 29].

(69)
$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^nq^{\binom{r}{2}+rM}}{\langle 1-\alpha+r;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}.$$

Proof.

(70)

$$LHS = \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n q^{\binom{r}{2}+rM}}{\langle 1-\alpha+r;q\rangle_M \langle 1;q\rangle_m \langle 1;q\rangle_n}$$

$$= \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha;q\rangle_{r+M}} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r} q^{r(-\alpha+\beta+M+N)}$$

$$= \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)}{\langle 1-\alpha;q\rangle_M} \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$

$$q^{r(-\alpha+\beta+M+N)} = \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)}{\langle 1-\alpha;q\rangle_M} \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha+M;q\rangle_N} =$$

$$\frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}.$$

□

Theorem 4.12. *Compare* [17, (9) p. 244].

(71)

$$\sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n q^{\binom{r}{2}+r(\alpha-\beta+1-N)}}{\langle 1-\alpha+r;q\rangle_M \langle 1;q\rangle_m \langle 1;q\rangle_n} =$$

$$\frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n q^{NM}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+N;q\rangle_M q^{N(1-\beta)}}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}.$$

Proof.

(72)

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^nq^{\binom{r}{2}+r(\alpha-\beta+1-N)}}{\langle 1-\alpha+r;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}$$

$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha;q\rangle_{r+M}}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r$$

$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle 1-\alpha;q\rangle_M}\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r}\frac{\langle -N;q\rangle_r q^r}{\langle 1;q\rangle_r}$$

$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle 1-\alpha;q\rangle_M}\frac{x^my^nq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha+M;q\rangle_N}=$$

$$\frac{\langle \beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^nq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}.$$

□

Specializing the previous formulas leads to the following formulas for two variables, where we have put

(73)

$$\theta(N;\alpha,\beta;r;q)\equiv(-1)^r\binom{N}{r}_q q^{\binom{r}{2}}\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)};\ \omega(N;\alpha,\beta;q)\equiv\frac{\langle \beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}.$$

We have assumed that $B=B'=G=G'$ in all formulas. The conditions for A and E are given separately in each case.

Theorem 4.13. *Compare* [17, (16) p. 246]*. We have assumed that* $A+2l=E$*.*

(74)

$$\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)\Phi_{E:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;1-\beta+r),(a):(b);(b')\\(e):(g);(g')\end{array}|q;xq^{l(N-r)},yq^{l(N-r)}\right]=$$

$$\omega(N;\alpha,\beta;q)\Phi_{E+2l:G}^{A+4l:B}\left[\begin{array}{c}\triangle(q;l;1-\beta,1+\alpha-\beta),(a):(b);(b')\\\triangle(q;l;1+\alpha-\beta-N),(e):(g);(g')\end{array}|q;x,y\right].$$

Proof. Put

$$C(m,n)=\frac{\langle (a);q\rangle_{m+n}\langle (b);q\rangle_m\langle (b');q\rangle_n}{\langle (e);q\rangle_{m+n}\langle (g);q\rangle_m\langle (g');q\rangle_n}q^{Nl(m+n)}\tag{75}$$

in (47). We have assumed that $k=l$. □

In the following proofs we use the value

$$C(m,n) = \frac{\langle (a);q\rangle_{m+n}\langle (b);q\rangle_m\langle (b');q\rangle_n}{\langle (e);q\rangle_{m+n}\langle (g);q\rangle_m\langle (g');q\rangle_n} \tag{76}$$

and assume that $k = l$.

Theorem 4.14. *Compare* [17, (19) p. 247]. *We have assumed that* $A = E$.

(77)
$$\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)\Phi_{E+2l:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;1-\beta+r),(a):(b);(b')\\ \triangle(q;l;1-\alpha+r),(e):(g);(g')\end{array}|q;x,y\right] =$$
$$\omega(N;\alpha,\beta;q)\Phi_{E+2l:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;1-\beta),(a):(b);(b')\\ \triangle(q;l;1-\alpha+N),(e):(g);(g')\end{array}|q;x,y\right].$$

Proof. Use (51). □

Theorem 4.15. *Compare* [17, (20) p. 247]. *We have assumed that* $A = E$.

(78)
$$\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)\Phi_{E+2l:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;\alpha-r),(a):(b);(b')\\ \triangle(q;l;\beta-r),(e):(g);(g')\end{array}|q;x,y\right] =$$
$$\omega(N;\alpha,\beta;q)\Phi_{E+2l:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;\alpha-N),(a):(b);(b')\\ \triangle(q;l;\beta),(e):(g);(g')\end{array}|q;xq^{Nl},yq^{Nl}\right].$$

Proof. Use (55). □

Theorem 4.16. *Compare* [17, (21) p. 247]. *We have assumed that* $A = E + 2l$.

(79)
$$\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)q^{r(1+\alpha-\beta-N)}\Phi_{E+2l:G}^{A:B}\left[\begin{array}{c}(a):(b);(b')\\ \triangle(q;l;\beta-r),(e):(g);(g')\end{array}|q;xq^{-rl},yq^{-rl}\right] =$$
$$\omega(N;\alpha,\beta;q)q^{N(1-\beta)}\Phi_{E+4l:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;\beta-\alpha+N),(a):(b);(b')\\ \triangle(q;l;\beta,-\alpha+\beta),(e):(g);(g')\end{array}|q;xq^{-Nl},yq^{-Nl}\right].$$

Proof. Use (62). □

Theorem 4.17. *Compare* [17, (22) p. 248]. *We have assumed that* $A+2l=E$.

(80)
$$\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)\Phi_{E:G}^{A+2l:B}\left[\begin{matrix}\triangle(q;l;\alpha-r),(a):(b);(b')\\(e):(g);(g')\end{matrix}|q;x,y\right]=$$
$$\omega(N;\alpha,\beta;q)\Phi_{E+2l:G}^{A+4l:B}\left[\begin{matrix}\triangle(q;l;\alpha-N,1+\alpha-\beta),(a):(b);(b')\\\triangle(q;l;1+\alpha-\beta-N),(e):(g);(g')\end{matrix}|q;x,y\right].$$

Proof. Use (64). □

Remark 6. In [14, (3.1) p. 439] Kandu tried to derive a similar formula. However, in this article the definitions are insufficient.

Remark 7. Formula (80) is a q-analogue of [12].

Theorem 4.18. *Compare* [17, (23) p. 248]. *We have assumed that* $A=E+2l$.

(81)
$$\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)\Phi_{E+2l:G}^{A:B}\left[\begin{matrix}(a):(b);(b')\\\triangle(q;l;1-\alpha+r),(e):(g);(g')\end{matrix}|q;xq^{rl},yq^{rl}\right]=$$
$$\omega(N;\alpha,\beta;q)\Phi_{E+4l:G}^{A+2l:B}\left[\begin{matrix}\triangle(q;l;-\alpha+\beta+N),(a):(b);(b')\\\triangle(q;l;1-\alpha+N,-\alpha+\beta),(e):(g);(g')\end{matrix}|q;x,y\right]$$

Proof. Use (69). □

5. Appendix

We show that the q-analogue of Burchnall-Chaundy [2, (43)] from [5] can't be used to produce another q-analogue of the reduction formula for the second Appell function. We know that

(82)
$$\Phi_2(a;b,b;c,c|q;x_1,x_2)=\sum_{r=0}^{\infty}(-1)^r\frac{\langle a;q\rangle_{2r}\langle b,c-b;q\rangle_r}{\langle c;q\rangle_{2r}\langle 1,c;q\rangle_r}q^{\binom{r}{2}+rb}{x_1}^r{x_2}^r\times$$
$${}_2\phi_1(a+2r,b+r;c+2r|q;x_1\oplus_q x_2).$$

In order to produce another q-analogue of the reduction formula for the second Appell function, we would have to change the $x_1\oplus_q x_2$ to $x_1\boxplus_q x_2$ and then put $x_2=-q^l x_1$. However this can't be done as the following computation shows:

(83)

$$\Phi^{1:2;1}_{1:1;1}\left[\begin{array}{c} a:b,\infty;b \\ \infty:c;c \end{array}|q;x_1,-x_2\right] = \nabla_q(c)\bigtriangleup_q(b)_2\phi_1(a,b;c|q;(x_1\boxplus_q x_2)) =$$

$$\sum_{r=0}^{\infty}\frac{\langle c-b,-\theta_{q,1},-\theta_{q,2};q\rangle_r}{\langle 1,c,1-b-\theta_{q,1}-\theta_{q,2};q\rangle_r}q^r\sum_{m,n=0}^{\infty}\frac{\langle a,b;q\rangle_{m+n}x_1^m x_2^n q^{\binom{n}{2}}}{\langle c;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1;q\rangle_n} =$$

$$\sum_{r=0}^{\infty}\frac{\langle c-b;q\rangle_r}{\langle 1,c;q\rangle_r}(-q)^r\sum_{m,n=r}^{\infty}\frac{\langle b;q\rangle_r\langle a,b-r;q\rangle_{m+n}x_1^m x_2^n q^{\binom{n}{2}+3\binom{r}{2}+ra-r^2}}{\langle b-r;q\rangle_{2r}\langle c;q\rangle_{m+n}\langle 1;q\rangle_{m-r}\langle 1;q\rangle_{n-r}} =$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\langle c-b,b;q\rangle_r\langle a;q\rangle_{2r}x_1^r x_2^r}{\langle 1,c;q\rangle_r\langle c;q\rangle_{2r}}q^{2\binom{r}{2}+ra}\sum_{m,n=0}^{\infty}\frac{\langle a+2r,b+r;q\rangle_{m+n}x_1^m x_2^n q^{\binom{n}{2}+nr}}{\langle c+2r;q\rangle_{m+n}\langle 1;q\rangle_m\langle 1;q\rangle_n} =$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r\langle c-b,b;q\rangle_r\langle a;q\rangle_{2r}x_1^r x_2^r}{\langle 1,c;q\rangle_r\langle c;q\rangle_{2r}}q_2^{2\binom{r}{2}+ra}\phi_1(a+2r,b+r;c+2r|q;(x_1\boxplus_q q^r x_2)).$$

6. Conclusion

In section 3, [5] we mainly considered double summation formulas of general terms, which could be specialized to reduction formulas for Kampé de Fériet functions. In this paper we found q-analogues of Buschman and Srivastava [3] double series identities and all the corresponding reduction formulas. We have tried to keep the order of equations in accordance with [3]. In the book [10] we have changed the notation to $\bigtriangleup(q;l;\lambda)$ whenever possible. Like in the previous papers, there was no need to recapitulate the hypergeometric formulas, a great advantage of our q-umbral method. Another advantage of this method is that we don't have to bother about the uncomfortable q-factors to the right of $||$, instead we can keep these factors after confluence. The important thing is that the exponent of q in the series remains positive, like in q^{mn}.

References

[1] Al-Salam W.A., Saalschützian theorems for basic double series. *J. London Math. Soc.* 1965 **40**, 455–458.

[2] Burchnall J.L. and Chaundy T.W., Expansions of Appell's double hypergeometric functions. *Quart. J. Math.*, Oxford Ser. 1940 **11**, 249–270.

[3] R. G.Buschman - Srivastava H.M., Series identities and reducibility of Kampé de Fériet functions. *Proc. Cambridge Philos. Soc.* 1982, **91**, no. 3, 435–440.

[4] Ernst T., A method for q-calculus. *J. nonlinear Math. Physics* 2003, **10** No.4, 487-525.

[5] Ernst T., Some results for q-functions of many variables. *Rendiconti di Padova* 2004,**112**, 199-235.

[6] Ernst T., q-Generating functions for one and two variables. *Simon Stevin* 2005, **12** no. 4, 589–605.

[7] Ernst T., A renaissance for a q-umbral calculus. Proceedings of the International Conference Munich, Germany 25 - 30 July 2005. World Scientific, 2007.

[8] Ernst T., Some new formulas involving Γ_q functions. Rendiconti di Padova 2007 **118**, 159–188.

[9] Ernst T., The different tongues of q-calculus, *Proceedings of Estonian Academy of Sciences.* 2008 **57** no. 2, 81-99.

[10] Ernst T., Handbuch für die q-Analysis, submitted.

[11] Gasper G. and Rahman M., *Basic hypergeometric series,* Cambridge, 1990.

[12] Gupta, K. C. and Srivastava A., Certain results involving Kampé de Fériet's function. *Indian J. Math.* 1973 **15**, 99–102.

[13] Jackson F.H., On basic double hypergeometric functions. *Quart. J. Math.*, Oxford Ser. 1942 **13**, 69–82.

[14] Kandu, D. Certain expansions involving basic hypergeometric functions of two variables. *Indian J. Pure Appl. Math.* 1987 **18**, no. 5, 438–441.

[15] MacRobert, T. M. The multiplication formula for the gamma function and E-function series. *Math. Ann.* 1959 **139**, 133–139 (1959).

[16] Panda R., Some multiple series transformations. *J nānabha Sect.* A 1974 **4** 165–168.

[17] Panda, R.; Srivastava, H. M. Some recursion formulas associated with multiple hypergeometric functions. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 1975 **23**, no. 3, 243–250.

[18] Rainville E. D., *Special functions.* Reprint of 1960 first edition. Chelsea Publishing Co., Bronx, N.Y., 1971.

[19] Shanker, O.; Saran, S. Reducibility of Kampé de Fériet function. *Ganita* 1970 no. 1, **21**, 9–16.

[20] Sharan, G. P. On summation and transformations of certain hypergeometric functions of three variables. *Publ. Inst. Math.* (Beograd) (N.S.) 1980 **27**(41), 225–227.

[21] Singhal B.M., On the reducibility of Lauricella's function F_D. *J nānabha* 1974 **A 4**, 163–164. 33A30

[22] Srivastava, H. M. Certain summation formulas involving generalized hypergeometric functions. *Comment. Math. Univ. St. Paul.* 1972/73 **21**, 25–34.

[23] Srivastava H.M., On the reducibility of Appell's function F_4. *Canad. Math. Bull.* 1973 **16**, 295–298.

[24] Srivastava H.M., A Watsonian theorem for multiple series. *Rend. Sem. Mat. Univ. Padova*, 1977 **58**, 241–245.

[25] Srivastava H.M., A note on certain identities involving generalized hypergeometric series. Nederl. Akad. Wetensch. *Indag. Math.*, 1979 **41** , no. 2, 191–201.

[26] Srivastava H.M.and Karlsson P.W., *Multiple Gaussian hypergeometric series.* Ellis Horwood, New York, 1985.

[27] H. M. Srivastava H.M. and Jain V. K., q-series identities and reducibility of basic double hypergeometric functions. *Canad. J. Math.* 1986 **38**, no. 1, 215–231.

[28] Srivastava H.M., Summation theorems for a certain class of multiple hypergeometric series. *Simon Stevin* 1984 **58**, 243-252.

[29] Srivastava H.M. , A class of finite q-series. *Rend. Sem. Mat. Univ. Padova* 1986 **75** ,15–24.

[30] Srivastava H.M. , A class of finite q-series. II. *Rend. Sem. Mat. Univ. Padova* 1986 **76** , 37–43.

[31] Srivastava H.M. and Karlsson P.W., Transformations of multiple q-series with quasi-arbitrary terms. [J] *J. Math. Anal. Appl.* 1999, **231**, No.1, 241-254.

Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden

E-mail address: `Thomas.Ernst@math.uu.se`