CONVERGENCE ASPECTS FOR q–APPELL FUNCTIONS I

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ABSTRACT. By using a certain q-Stirling approximation, we show that the convergence region for the second q–Appell function Φ_2 is formally decided by a certain Nalli-Ward-AlSalam (NWA) q-addition. The convergence region for Φ_1 is the same as for $q = 1$. In the process we investigate numerical aspects, including a local maximum, of the NWA q-addition of two letters.

1. INTRODUCTION

P.Appell and J.Kampé de Fériet [1], together with Lauricella introduced hypergeometric functions of many variables. The purpose of this paper, the first in a series of four, is to investigate convergence regions for q–Appell and q–Lauricella-functions presented in [4] and [6]. In [4] and [6] we found many summation, transformation, and reduction formulas, but convergence regions for the respective multiple q-functions were not given. It turns out that the crucial step towards finding these regions is the Nalli-Ward-AlSalam q-addition; this will be dealt with in section 2.

By using the q-Stirling formula, we find that the convergence regions for the q–Appell functions Φ_2, Φ_4 and the q–Lauricella-functions $\Phi_A^{(n)}, \Phi_C^{(n)}$ are, formally, decided by the same formulas as for $q = 1$, but with NWA. We illustrate with a comparison beteeen numerical values for NWA (a function called $F(n)$) and the corresponding convergence for $\Phi_2(x, x)$. Because of lack of space, we mostly compute function values for the two-variable functions F and Φ_2 on the diagonal; this should be sufficient, since the absolute value of these two functions increase in both variables. Another interesting property is the fact that $F(n)$ empirically always has a local maximum before turning to zero; $|\Phi_2|$ has a corresponding local minimum in q. The convergence region for the q–Kampé de Fériet function is also probably depending on NWA, but the proof of this is relegated to a later paper.

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This paper is organized as follows: In this first section we give a general overview. In section two we present numerical values for the function $F(n)_{a,b,q}$. In section 3 we present the q-Stirling proofs for the convergence regions of Φ_1 and Φ_2 together with numerical function values, which vindicate the theoretical assumptions.

2. The function $F(n)_{a,b,q}$

The following function will be crucial in this and the following paper:

Definition 1. The function $F(n)_{a,b,q}$ is defined by

$$F(n)_{a,b,q} = (a \oplus_q b)^n, 0 < a < 1, 0 < b < 1, 0 < q < 1. \tag{1}$$

For certain values of q, Mathematica computations show that $F(n)_{a,b,q}$ first increases and then decreases to zero. The following table shows the value of n for the local maximum of the function $F(n)_{a,a,.88}$ together with the value of n for the function value 1 (there are two such n since the function value 1 is usually not obtained). The function value 1 will be important for the convergence of multiple q-series later.

$a = b$	n for Maximum value of f	n for value $= 1$
.55	4	$7, 8$
.6	7	$14, 15$
.65	11	$23, 24$
.7	14	$32, 33$
.8	24	$62, 63$

We will denote some of the ensuing tables by capitals since we will later refer to them. The values of the parameters x_i and q are arbitrary, but other values will exhibit a similar behaviour, illustrating the monotonic behaviour of $F(n)$.

The following two tables (A) show the function values $F(n)_{.55,.55,.999}$:

n	$F(n)_{.55,.55,.999}$	n	$F(n)_{.55,.55,.999}$
1	1.1	201	1.49123×10^6
11	2.81418	211	2.3444×10^6
21	7.0223	221	3.59941×10^6
31	17.0924	231	5.39721×10^6
41	40.5834	241	7.90446×10^6
51	94.0035	251	1.13075×10^7
61	212.431	261	1.58009×10^7
71	468.376	271	2.15698×10^7
81	1007.63	281	2.87661×10^7
91	2115.29	291	3.74816×10^7
101	4333.35	301	4.7718×10^7
111	8663.42	311	5.9361×10^7
121	16904.2	321	7.2161×10^7
131	32193.6	331	8.5726×10^7
141	59846.2	341	9.9531×10^7
151	108599.	351	1.12945×10^8
161	192382.	361	1.25275×10^8
171	332719.	371	1.35825×10^8
181	561816.	381	1.43959×10^8
191	926276.	391	1.49166×10^8

n	$\mathrm{F(n)}_{.55,.55,.999}$
401	1.51111×10^8
411	1.49675×10^8
421	1.44961×10^8
431	1.37288×10^8
441	1.2715×10^8
451	1.15169×10^8
461	1.02027×10^8
471	8.84047×10^7
481	7.49291×10^7
491	6.21247×10^7
501	5.15068×10^7
511	4.09625×10^7
521	3.18732×10^7
531	2.42666×10^7
541	1.80784×10^7
551	1.31799×10^6
561	9.40338×10^6
571	6.56609×10^6
581	4.48753×10^6
591	3.002×10^6

The function $F(n)_{.55,.55,.999}$ has a local maximum 1.511196514538803 $\times 10^8$ for $n = 402$.

The function $F(n)_{.55,.55,.999}$ converges to zero for large values of N.

The following table (B) shows some function values $F(n)_{.66,.66,.99}$. The function $F(n)_{.66,.66,.99}$ has a local maximum 3.38441 $\times 10^7$ for $n =$

130.

n	$\text{F(n)}_{.66,.66,.99}$
110	2.1529×10^7
120	2.94974×10^7
130	3.38441×10^7
140	3.27045×10^7
250	1250.55
260	231.851
270	38.6279
280	5.80749
290	0.791094
300	0.0980218
600	1.34693×10^{-41}

The following table of function values $F(n)_{.66,.66,.999}$ shows that the estimate $q = .99$ from the previous table is pretty sharp.

n	$\text{F(n)}_{.66,.66,.999}$
220	9.07711×10^{23}
230	8.44739×10^{24}
240	7.67822×10^{25}
250	6.81692×10^{26}

The following table (C) shows some function values $F(n)_{.78,.78,.9}$:

n	$\text{F(n)}_{.78,.78,.9}$
599	9.82785×10^{-57}
600	7.67976×10^{-57}
601	6.00117×10^{-57}

The following table (D) shows some function values $F(n)_{.8,.8,.9665}$. The function $F(n)_{.8,.8,.9665}$ has a local maximum 6.55105 $\times 10^6$ for $n = 110$.

n	$F(n)_{.8,.8,.9665}$	n	$F(n)_{.8,.8,.9665}$
100	1.20969×10^7	360	1.13583×10^{-13}
110	6.55105×10^6	370	1.31232×10^{-14}
120	2.83086×10^6	380	1.50914×10^{-15}
130	1.00705×10^6	390	1.72823×10^{-16}
140	303109.	400	1.97174×10^{-17}
150	79053.2	410	2.24202×10^{-18}
160	18238.9	420	2.54165×10^{-19}
170	3789.66	430	2.87344×10^{-20}
180	720.112	440	3.24048×10^{-21}
190	126.802	450	3.64613×10^{-22}
200	20.9256	460	4.09408×10^{-23}
210	3.26764	470	4.58833×10^{-24}
220	0.486799	480	5.13327×10^{-25}
230	0.0696703	490	5.73367×10^{-26}
240	0.00963595	500	6.39474×10^{-27}
250	0.00129438	510	7.12216×10^{-28}
260	0.000169587	520	7.9221×10^{-29}
270	0.0000217493	530	8.80132×10^{-30}
280	2.73863×10^{-6}	540	9.76713×10^{-31}
290	3.3945×10^{-7}	550	1.08275×10^{-31}
300	4.15061×10^{-8}	560	1.19912×10^{-32}
310	5.01584×10^{-9}	570	1.32677×10^{-33}
320	5.99991×10^{-10}	580	1.46672×10^{-34}
330	7.11361×10^{-11}	590	1.6201×10^{-35}
340	8.36888×10^{-12}	600	1.78812×10^{-36}
350	9.77892×10^{-13}	610	

The purpose of the tables E and F is to show the decreasing property of F for the respective parameter values.

The following table (E) shows some function values $F(n)_{.97,.97,.89}$:

n	$F(n)_{.97,97,.89}$
599	1.20525
600	1.1712
601	1.13812

The following table (F) shows some function values $F(n)_{.97,.97,.899}$:

n	$\mathrm{F(n)}_{.97,97,.899}$
599	4.32638
600	4.20426
601	4.08558

The next two tables show in detail the behaviour of $F(n)$ for consecutive n and certain parameter values. The table G_1 shows some function values $F(n)_{.985,.985,.895}$:

n	$\mathrm{F(n)}_{.985,.985,.895}$	n	$\mathrm{F(n)}_{.985,.985,.895}$
1	1.97	2	3.77903
3	7.06437	4	12.8789
5	22.9152	6	39.8233
7	67.646	8	112.398
9	182.814	10	291.275
11	454.934	12	697.031
13	1048.36	14	1548.89
15	2249.39	16	3213.14
17	4517.38	18	6254.7
19	8533.96	20	11480.9
21	15237.9	22	19964.
23	25832.7	24	33031.2
25	41757.	26	52215.1
27	64614.7	28	79165.1
29	96071.4	30	115531.
31	137728.	32	162831.
33	190990.	34	222329.
35	256950.	36	294925.
37	336299.	38	381086.
39	429272.	40	480813.
41	535637.	42	593648.
43	654721.	44	718712.
45	785458.	46	854776.
47	926471.	48	1.00034×10^6
49	1.07615×10^6	55	1.55947×10^6

The G_2 shows some function values $F(n)_{.985,.985,.895}$:

n	$F(n)_{.985,.985,.895}$	n	$F(n)_{.985,.985,.895}$
60	1.97542×10^6	70	2.75086×10^6
80	3.36878×10^6	90	3.80027×10^6
100	4.05999×10^6	110	4.17762×10^6
120	4.18427×10^6	130	4.10749×10^6
140	3.97013×10^6	150	3.79056×10^6
160	3.58326×10^6	170	3.35955×10^6
180	3.1281×10^6	190	2.89547×10^6
200	2.66656×10^6	210	2.4449×10^6
230	2.03237×10^6	250	1.66869×10^6
270	1.35645×10^6	290	1.09356×10^6
310	875523.	330	696824.
350	551778.	370	434987.
390	341581.	410	267307.

The function $F(n)_{.985,.985,.895}$ has a local maximum 4.19306×10^6 for $n = 116$.

The following table (H) shows some function values $F(n)_{.99,.99,.895}$:

n	$F(n)_{.99,.99,.895}$	n	$F(n)_{.99,.99,.895}$
210	7.08017×10^6	230	6.51275×10^6
250	5.91721×10^6	270	5.32261×10^6
290	4.74835×10^6	310	4.20675×10^6
330	3.70494×10^6	350	3.2464×10^6
370	2.832×10^6	390	2.46088×10^6
410	2.13101×10^6		

The function $F(n)_{.99,.99,.895}$ has a local maximum 8.10154×10^6 for $n = 149$.

The following table (I) shows some function values $F(n)_{.995,.995,.895}$:

n	$F(n)_{.995,.995,.895}$	n	$F(n)_{.995,.995,.895}$
410	1.6812×10^7	430	1.60521×10^7
450	1.52843×10^7	470	1.45169×10^7
490	1.37568×10^7	510	1.30096×10^7
530	1.22798×10^7	550	1.15709×10^7
570	1.08854×10^7	590	1.02255×10^7
610	9.59238×10^6		

The function $F(n)_{.995,.995,.895}$ has a local maximum 2.08494×10^7 for $n = 249$.

The function $F(n)_{.996,.996,.895}$ has a local maximum 2.74055×10^7 for $n = 299$.

Finally we show some function values for $F(n)_{.999,.999,.895}$:

n	$F(n)_{.999,.999,.895}$	n	$F(n)_{.999,.999,.895}$
760	1.20952×10^8	800	1.2275×10^8
900	1.25863×10^8	1000	1.2727×10^8
1200	1.26114×10^8		

Obviously the function $F(n)_{.999,.999,.895}$ has a local maximum between $n = 900$ and $n = 1200$.

3. q-Appell functions

In this section we study asymptotic expansions of q-Appell functions. We start with the q-Stirling formula.

3.1. **The q-Stirling formula.** There is a plethora of Stirling's (and similar) formulas in the literature, which describe the asymptotic behavior of the gamma function. The following form is most often used:

$$\Gamma(\lambda + k) \sim k^{\lambda+k-\frac{1}{2}} e^{-k} \sqrt{2\pi}. \tag{2}$$

Previously there were two versions of the q-Stirling formula [2] and [11]. Both formulas were written in logarithmic form, with the same first term. We will however use the following equivalent approximation:

$$\Gamma_q(z) \sim \{z\}_q^{\,z-\frac{1}{2}}. \tag{3}$$

3.2. **Computations.** All the computations in this and the following section are q-analogues of Exton [7, p. 25-26].

Definition 2. The q-analogues of the two first Appell functions are

$$\Phi_1(a; b, b'; c|q; x_1, x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a; q\rangle_{m_1+m_2} \langle b; q\rangle_{m_1} \langle b'; q\rangle_{m_2}}{\langle 1; q\rangle_{m_1} \langle 1; q\rangle_{m_2} \langle c; q\rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}. \tag{4}$$

$$\Phi_2(a;b,b';c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{5}$$

Sometimes we wish to emphasize the partial sum $\sum_{m_1,m_2=0}^{N}$ of these functions, we denote this partial sum by $\Phi_1(N;a;b,b';c|q;x_1,x_2)$ etc.

Consider $\Phi_1(a;b,b';c|q;x_1,x_2)$. The coefficient of $x_1^m x_2^n$ is equal to

$$A_{m,n} \equiv \Gamma_q \left[\begin{array}{c} c, a+m+n, b+m, b'+n, 1, 1 \\ a, b, b', c+m+n, 1+m, 1+n \end{array} \right]. \tag{6}$$

According to the $q-$Stirling formula [2, p. 899], $\lim_{m,n\to\infty}$

$$A_{m,n} \sim \Gamma_q \left[\begin{array}{c} c \\ a, b, b' \end{array} \right] \lim_{m,n\to\infty} \{m+n\}_q^{a-c}\{m\}_q^{b-1}\{n\}_q^{b'-1}. \tag{7}$$

The real parts of a,b,b',c are $a_1, b,_1\, b'_1, c_1$ and N is a number such that

$$N > \left| \Gamma_q \left[\begin{array}{c} c \\ a, b, b' \end{array} \right] \right| \tag{8}$$

For m,n big enough, we have

$$|A_{m,n}x_1^m x_2^n| < N|x_1|^m|x_2|^n\{m+n\}_q^{a_1-c_1}\{m\}_q^{b_1-1}\{n\}_q^{b'_1-1}. \tag{9}$$

The function Φ_1 converges for $|x_1| < 1$, $|x_2| < 1$.

We illustrate with an example for $x_2 = 1$.

N	$\Phi_1(N;.7;.66;.65,.44\|.9;.99,1.)$
100	1.64857×10^7
120	2.17317×10^7
140	2.7177×10^7

Since $\triangle\Phi_1 \approx .5 \times 10^7$ no convergence is in sight, which is in accordance with our assumption.

Consider $\Phi_2(a;b,b';c,c'|q;x_1,x_2)$. The coefficient of $x_1^m x_2^n$ is equal to

$$A_{m,n} \equiv \Gamma_q \left[\begin{array}{c} c, c', a+m+n, b+m, b'+n, 1, 1 \\ a, b, b', c+m, c'+n, 1+m, 1+n \end{array} \right]. \tag{10}$$

According to the $q-$Stirling formula, $\lim_{m,n\to\infty}$

$$A_{m,n} \sim \Gamma_q \left[\begin{array}{c} c, c' \\ a, b, b' \end{array} \right] \lim_{m,n\to\infty} \{m+n\}_q^{a-1}\{m\}_q^{b-c}\{n\}_q^{b'-c'} \binom{m+n}{m}_q \tag{11}$$

The real parts of a, b, b', c, c' are $a_1, b_1, b'_1, c_1, c'_1$, and N is a number such that

$$N > \left| \Gamma_q \left[\begin{array}{c} c, c' \\ a, b, b' \end{array} \right] \right| \tag{12}$$

For m, n big enough, we have

$$|A_{m,n} x_1^m x_2^n| < N \binom{m+n}{n}_q \{m+n\}_q^{a_1-1} \{m\}_q^{b_1-c_1} \{n\}_q^{b'_1-c'_1} |x_1|^m |x_2|^n. \tag{13}$$

If ϵ denotes a positive number bigger than the greatest of $b_1 - c_1$ and $b'_1 - c'_1$ and ϵ' is a sufficiently big number,

$$\{m\}_q^{b_1-c_1} \{n\}_q^{b'_1-c'_1} < \{m\}_q^{\epsilon} \{n\}_q^{\epsilon} < \frac{\{m+n\}_q^{2\epsilon'}}{4^\epsilon}. \tag{14}$$

Therefore

$$\sum_{m,n=0}^{\infty} |A_{m,n} x_1^m x_2^n| < \frac{N}{4^\epsilon} \sum_{m,n=0}^{\infty} \binom{m+n}{n}_q \{m+n\}_q^{2\epsilon'+a_1-1} |x_1|^m |x_2|^n < \frac{N}{4^\epsilon} \sum_{r=0}^{\infty} \{r\}_q^{2\epsilon'+a_1-1} \left(|x_1| \oplus_q |x_2|\right)^r, \tag{15}$$

where $r = m + n$ and the series converges for $(|x_1| \oplus_q |x_2|)^r < 1$ and Φ_2 converges in the same region.

For q=1 the convergence region for Φ_2 is a rhombus bounded by lines like $x_2 = 1 - x_1$. In our case, the convergence region is q-dependent and the farer away q is below 1 ($0 < q < 1$), the greater convergence region.

3.3. **Numerical values.** We have collected some approximate function values in the following tables to show the extended convergence region. Although the function values are quite large for x_1, x_2 close to 1, we can still count on convergence.

The first table shows the extended convergence region for the specific value $q = .88$, compare with table C.

x	$\Phi_2(.7; .66, -.88; .65, .44|.88; x, x)$
.51	-1.12541
.53	-1.42309
.55	-1.77242
.57	-2.18652
.59	-2.68332
.61	-3.28788
.63	-4.0363
.65	-4.98204
.67	-6.20696
.69	-7.84093
.71	-10.0978
.73	-13.3442
.74	-15.5261
.75	-18.2363
.77	-26.0028
.79	-39.0605
.81	-62.4227
.83	-107.119
.85	-199.096
.87	-404.343
.89	-907.399
.91	-2290.84
.92	-3840.02
.93	-6725.43
.935	-9074.36
.94	-12428.8
.945	-17316.8
.95	-24609.7
.955	-35800.4
.96	-53567

The following three tables show the same trend of extended convergence region for mixed values of q and x.

q	x	$\Phi_2(.7;.66,-.88;.65,.44\|q;x,x)$
.5	.66	-5.61604
.51	.67	-5.96231
.52	.68	-6.34313
.53	.69	-6.76323
.54	.7	-7.22824
.55	.71	-7.74484
.56	.72	-8.32109
.57	.73	-8.96674
.58	.74	-9.6937
.59	.75	-10.5166
.6	.76	-11.4538
.61	.77	-12.5282
.62	.78	-13.7691
.63	.79	-15.2143
.64	.8	-16.9133
.65	.81	-18.9321
.66	.82	-21.3598
.67	.83	-24.3198
.68	.84	-27.9866
.69	.85	-32.6127

q	x	$\Phi_2(.7;.66,-.88;.65,.44\|q;x,x)$
.6	.71	-7.48447
.61	.72	-8.08783
.62	.73	-8.7679
.63	.74	-9.53886
.64	.75	-10.4185
.65	.76	-11.4293
.66	.77	-12.6005
.67	.78	-13.9697
.68	.79	-15.5872
.69	.8	-17.5208
.7	.81	-19.864
.71	.82	-22.7485
.72	.83	-26.365
.73	.84	-30.9966
.74	.85	-37.0783
.75	.86	-45.302
.76	.87	-56.8143
.77	.88	-73.6037
.78	.89	-99.3057
.79	.9	-140.98
.8	.91	-213.328
.81	.92	-349.595
.82	.93	-632.674
.83	.94	-1295.38
.84	.95	-3095.79
.85	.96	-9021.62
.86	.97	-34269.8
.87	.98	-188906

q	x	$\Phi_2(.7;.66,-.88;.65,.44\vert q;x,x)$
.66	.82	−21.3598
.67	.83	−24.3198
.68	.84	−27.9866
.69	.85	−32.6127
.7	.86	−38.5743
.71	.87	−46.4494
.72	.88	−57.1591
.73	.89	−72.2303
.74	.90	−94.3171
.75	.91	−128.288
.76	.92	−183.661
.77	.93	−280.511
.78	.94	−465.292
.79	.95	−858.829
.8	.96	−1827.69
.81	.97	−4754.61
.82	.98	−17013.4
.83	.99	−114877.

In the beginning of the following table a partial sum to 50 is enough, while $N = 300$ is used at the end. The trend is the same as before.

q	x	$\Phi_2(.7;.66,-.88;.65,.44\|q;x,x)$
.7	.71	-7.36621
.71	.72	-8.07539
.72	.73	-8.89294
.73	.74	-9.84464
.74	.75	-10.9651
.75	.76	-12.3019
.76	.77	-13.9219
.77	.78	-15.922
.78	.79	-18.447
.79	.80	-21.7218
.8	.81	-26.1098
.81	.82	-32.2268
.82	.83	-41.1752
.83	.84	-55.0539
.84	.85	-78.1605
.85	.86	-120.069
.86	.87	-204.356
.87	.88	-396.52
.88	.89	-907.399
.89	.90	-2552.9
.9	.91	-9330.26
.91	.92	-47976.7
.92	.93	-392592

q	x	$\Phi_2(.7; .66, -.88; .65, .44\|q; x, x)$
.8	.66	−4.93827
.8	.71	−7.96101
.81	.67	−5.46816
.81	.72	−8.97323
.82	.68	−6.0907
.82	.73	−10.2226
.83	.69	−6.83527
.83	.74	−11.8086
.84	.7	−7.7462
.84	.75	−13.8951
.85	.71	−8.89392
.85	.76	−16.7687
.86	.72	−10.3968
.86	.77	−20.966
.87	.73	−12.4683
.87	.78	−27.5791
.88	.74	−15.5261
.88	.79	−39.0605
.89	.75	−20.4744
.89	.80	−61.6286

For the first value in the next table one should refer to table C.

q	x	$\Phi_2(.7; .66, -.88; .65, .44\|q; x, x)$
.90	.76	−29.5305
.90	.81	−113.63
.90	.93	−32819.6
.91	.77	−49.0653
.91	.82	−260.854
.91	.9	−13243.8
.92	.67	−7.61002
.92	.78	−101.625
.92	.83	−809.32
.92	.89	−22263.1
.93	.66	−7.1667
.93	.79	−293.432
.93	.84	−3805.06
.93	.88	−47212.9

For the last value in the next table one should refer to table B ($.65 < .66$).

q	x	$\Phi_2(.7; .66, -.88; .65, .44\|q; x, x)$
.94	.65	−6.8093
.94	.73	−51.5876
.94	.8	−1383.17
.94	.85	−32655.8
.94	.87	−140857
.95	.64	−6.56898
.95	.69	−22.5995
.95	.72	−71.2905
.95	.81	−13809.6
.95	.86	−722168
.96	.63	−6.52789
.96	.68	−29.9907
.96	.71	−129.497
.96	.76	−3347.17
.96	.81	−198592
.96	.82	−495698
.97	.62	−6.95628
.97	.67	−56.6479
.97	.70	−420.628
.97	.75	−30155.5
.98	.61	−9.31541
.98	.66	−290.289
.98	.69	−6128.56
.99	.60	−51.6401
.99	.65	−100577

We now consider $\Phi_2(.7; .66, -.88; .65, .44|q; x, x)$ as a function of q for fixed x. We will see that there is a minimum of $|\Phi|$ for a certain q_x and that $|\Phi|$ turns to ∞ when q is close to 1 ($x > .5$). The trend of inceasing converegence region is the same as before.

We have often divided the results into two consecutive tables with the same value of x; in each of these cases, the first table illustrates the behaviour near the minimum, and the second table illustrates the behaviour for larger q-values.

q	$\Phi_2(.7; .66, -.88; .65, .44 \vert q; .55, .55)$
.76	−1.82199
.77	−1.81156
.78	−1.80217
.79	−1.79385
.8	−1.7866
.81	−1.78045
.82	−1.77543
.83	−1.77158
.84	−1.76894
.85	−1.76761
.86	−1.76766
.87	−1.76921
.88	−1.77242

We have minimum 1.7676 of $|\Phi|$ for $q = .85$. For the last but one value in the next table one should refer to table A.

q	$\Phi_2(.7; .66, -.88; .65, .44 \vert q; .55, .55)$
.981	−2.11519
.982	−2.13524
.983	−2.15773
.984	−2.18323
.985	−2.21248
.986	−2.24649
.987	−2.28672
.988	−2.33529
.989	−2.39543
.99	−2.47232
.991	−2.57479
.992	−2.71908
.993	−2.93833
.994	−3.3102
.995	−4.05709
.996	−6.0494
.997	−15.2277
.998	−178.212
.999	−451960
.99904	−765139.

q	$\Phi_2(.7; .66, -.88; .65, .44 \vert q; .6, .6)$
.67	−2.98946
.68	−2.96909
.69	−2.95046
.7	−2.93354
.71	−2.91833
.72	−2.90482
.73	−2.89302
.74	−2.88295
.75	−2.87461
.76	−2.86806
.77	−2.86333
.78	−2.86049
.79	−2.85961
.8	−2.8608
.81	−2.86418
.82	−2.86991
.83	−2.8782
.84	−2.8893
.85	−2.90354
.86	−2.92135
.87	−2.94331
.88	−2.97019
.89	−3.00306

We have minimum 2.85961 of $|\Phi|$ for $q = .79$. For the last value in the next table one should refer to table B.

q	$\Phi_2(.7;.66,-.88;.65,.44\vert q;.6,.6)$
.981	-6.80324
.982	-7.41361
.983	-8.21458
.984	-9.29742
.985	-10.814
.986	-13.0304
.987	-16.4416
.988	$-22.0393,$
.989	-31.996
.99	-51.6401
.991	-96.0675
.992	-217.038
.993	-645.255
.994	-2885.24
.995	-24743.3
.996	-665800

q	$\Phi_2(.7;.66,-.88;.65,.44\vert q;.66,.66)$
.5	-5.61604
.65	-4.89325
.66	-4.87204
.67	-4.85395
.68	-4.83897
.69	-4.82714
.7	-4.8185
.71	-4.81312
.72	-4.8111
.73	-4.81255
.74	-4.81762
.75	-4.82649

We have minimum 4.8111 of $|\Phi|$ for $q = .72$.

q	$\Phi_2(.7;.66,-.88;.65,.44\vert q;.66,.66)$
.8	-4.93827
.93	-7.1667
.98	-290.289
.986	-7284.94
.987	-17137.9
.988	-46897.3
.989	-155379
.99	$-656610.$

q	$\Phi_2(.7;.66,-.88;.65,.44\vert q;.8,.8)$
.57	-16.862
.58	-16.8212
.59	-16.7957
.6	-16.7859
.61	-16.7921
.62	-16.815
.63	-16.8551
.64	-16.9133
.69	-17.5208
.79	-21.7218
.89	-61.6286

We have minimum 16.7859 of $|\Phi|$ for $q = .6$. For the last value in the next table one should refer to table D.

q	$\Phi_2(.7;.66,-.88;.65,.44\vert q;.8,.8)$
.965	$-503920.$
.9655	$-622634.$
.966	$-774293.$
.9665	$-969395.$

q	$\Phi_2(300;.7;.66,-.88;.65,.44\vert q;.8,.8)$
.925	110256.
.9255	116352.
.926	122870.
.9265	129846.
.927	137319.
.9275	145329.

We now make a leap to $x = .97$ and larger values.

q	$\Phi_2(.7; .66, -.88; .65, .44\|q; .97, .97)$
.51	-464.587
.52	-464.552
.53	-465.389
.54	-467.143

We have minimum 464.552 of $|\Phi|$ for $q = .52$. For the two last values in the next table one should refer to tables E,F. We use a partial sum $N = 880$.

q	$\Phi_2(.7; .66, -.88; .65, .44\|q; .97, .97)$
.81	-4754.61
.86	-34269.8
.871	-68705.6
.872	-73640.1
.873	-79026.9
.874	-84915.5
.875	-91361.6
.876	-98428.2
.877	$-106186.$
.878	$-114716.$
.879	$-124109.$
.880	$-134469.$
.881	$-145913.$
.882	$-158577.$
.883	$-172613.$
.884	$-188198.$
.885	$-205534.$
.886	$-224852.$
.887	$-246419.$
.888	$-270545.$
.889	$-297585.$
.890	$-327955.$
.899	$-865181.$

In the next table we show that the convergence area is not changed very much if we change one of the parameters.

q	$\Phi_2(.7; .66, -.88; .65, .34\|q; .97, .97)$
.871	$-108513.$

q	$\Phi_2(.7;.66,-.88;.65,.44\|q;.98,.98)$
.5	-753.76
.51	-752.66
.52	-752.991
.53	-754.818
.54	-758.227

We have minimum 752.6608 of $|\Phi|$ for $q=.51$.

We continue our investigation with $x=.98$.

q	$\Phi_2(.7;.66,-.88;.65,.44\|q;.98,.98)$
.82	-17013.4
.87	-188906

For the following table one should compare with the tables G_1, G_2.

q	$\Phi_2(280;.7;.66,-.88;.65,.44\|q;.98,.98)$
.891	-1.21996×10^6
.892	-1.35729×10^6
.893	-1.51339×10^6
.894	-1.69123×10^6
.895	-1.91513×10^6

For the following table one should compare with the tables G_2, H and I.

x	$\Phi_2(.7;.66,-.88;.65,.44\|.895;x,x)$	x	$\Phi_2(.7;.66,-.88;.65,.44\|.895;x,x)$
.985	-4.27×10^6	.99	-1.2×10^7
.995	-6.03×10^7		

q	$\Phi_2(.7;.66,-.88;.65,.44\|q;.99,.99)$
.49	-1319.77
.5	-1315.82
.51	-1314.4
.52	-1315.61
.53	-1319.57

We have minimum 1314.4 of $|\Phi|$ for $q=.51$.

Compare with table H.

q	$\Phi_2(260; .7; .66, -.88; .65, .44\|q; .99, .99)$
.85	$-257382.$
.879	-1.91729×10^6
.88	-2.09552×10^6
.881	-2.29412×10^6

The following two tables show that the q-analogues of the second Appell function for small values of the two variables are quite close to the ordinary values, which we have taken from [10, p. 355]

x	y	$F_2(.5, .5, 1, 1, 2; x, y)$
-0.02	-0.02	.9902
-0.01	-0.01	.9950
-0.004	-0.004	.9980
-0.02	-0.08	.9761
-0.01	-0.04	.9878
-0.005	-0.02	.9938

q	x	y	$\Phi_2(q; .5, .5, 1, 1, 2\|q; x, y)$
.8	-0.02	-0.02	0.988783
.9	-0.02	-0.02	0.989529
.8	-0.01	-0.01	0.994337
.9	-0.01	-0.01	0.994716
.8	-0.004	-0.004	0.997721
.9	-0.004	-0.004	0.997874
.8	-0.02	-0.08	0.972393
.9	-0.02	-0.08	0.974371
.8	-0.01	-0.04	0.985851
.9	-0.01	-0.04	0.986883
.8	-0.005	-0.02	0.992835
.9	-0.005	-0.02	0.993363

In [6] the following reduction formula for the second Appell function was found. For the notation, see [6].

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{\langle\lambda;q\rangle_{m+n}x^{m+n}(-1)^n\langle g;q\rangle_m\langle g,\widetilde{1-\tfrac{m+n}{2}};q\rangle_n\mathrm{QE}\left(\tfrac{-mn-n}{2}\right)}{\langle 1,h;q\rangle_m\langle 1,h,\widetilde{-\tfrac{m+n}{2}};q\rangle_n}\\&= {}_7\phi_7\left[\begin{matrix}\triangle(q;2;\lambda),g,h-g\\\triangle(q;2;h),h,\widetilde{1},\infty\end{matrix}|q;-x^2q^g||\begin{matrix}\langle\widetilde{1-k};q\rangle_k\\-\end{matrix}\right].\end{aligned}\tag{16}$$

The previous proof for the convergence area for the LHS of (16) works here as well, which is born out by Mathematica computations.

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