CONVERGENCE ASPECTS FOR q–APPELL FUNCTIONS II

THOMAS ERNST

Abstract By using a certain q-Stirling approximation, we show that the convergence region for the fourth q–Appell function Φ_4 is formally decided by a certain NWA q-addition. The convergence region for Φ_3 is the same as for $q = 1$.
Keywords q-Stirling formula, NWA q-addition, convergence region.

1. Introduction

We continue the study of q–Appell and q–Lauricella functions from [4], [7] and [8]. In this paper we concentrate on the third and fourth q–Appell function. This article, the second in a series of four, introduces a new convergence concept for (q)-functions of many variables. The reason is that the coefficients in the multiple power series are not constant. The key concept is the Ward q-addition, to be introduced shortly. This paper is organised as follows: In this section, we give a general introduction. In section two we present numerical values for the function $F(n)$ and present the q-Stirling proofs for the convergence regions of Φ_3 and Φ_4 together with a few numerical function values, which vindicate the theoretical assumptions.

We start with the definitions, compare with the book [9].

Definition 1. The power function is defined by $q^a \equiv e^{alog(q)}$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane. The variables $a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$ denote parameters in hypergeometric series or q-hypergeometric series. The variables
i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary

Date: March 20, 2012.
[0]2010 Mathematics Subject Classification: Primary 33C65; Secondary 33D70

unit. The q-shifted factorial is defined by:

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots, \end{cases} \tag{1}$$

In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod$\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R}\times e^{2\pi i\theta},\ \theta\in\mathbb{R}$. The operator

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{2}$$

Furthermore we define

$$\widetilde{\langle a;q\rangle_n} \equiv \langle \widetilde{a};q\rangle_n. \tag{3}$$

It follows that

$$\widetilde{\langle a;q\rangle_n} = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{4}$$

In equations (5) to (6) we assume that $(m,l)=1$. The operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l\log q}. \tag{5}$$

This means

$$\langle \widetilde{\tfrac{m}{l}} a;q\rangle_n = \prod_{m=0}^{n-1}(1-e^{2\pi i\frac{m}{l}}q^{a+m}), \tag{6}$$

The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q = \frac{1-q^a}{1-q},\ q\in\mathbb{C}\backslash\{1\}, \tag{7}$$

$$\{n\}_q! = \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q!=1,\ q\in\mathbb{C}, \tag{8}$$

Let the q-binomial coefficients be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}}. \tag{9}$$

Let a and b be any elements with commutative multiplication. Then the Nalli-Ward-AlSalam (NWA) q-addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n = 0, 1, 2, \ldots. \tag{10}$$

To justify the following definition of an infinite product we remind the reader of the following well–known theorem from complex analysis, see Rudin [14, p. 300]:

Theorem 1.1. *Let Ω be a region in the complex plane and let $H(\Omega)$ denote the holomorphic functions in Ω. Suppose $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots,$ no f_n is identically 0 in any component of Ω, and*

$$\sum_{n=1}^{\infty} |1 - f_n(z)| \tag{11}$$

converges uniformly on compact subsets of Ω. Then the product

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \tag{12}$$

converges uniformly on compact subsets of Ω. Hence $f \in H(\Omega)$.

Definition 2. The following function is holomorphic:

$$\langle a; q\rangle_\infty \equiv \prod_{m=0}^{\infty} (1 - q^{a+m}),\ 0 < |q| < 1. \tag{13}$$

The q-gamma function is defined by

$$\Gamma_q(x) \equiv \frac{\langle 1; q\rangle_\infty}{\langle x; q\rangle_\infty}(1 - q)^{1-x},\ 0 < q < 1. \tag{14}$$

The following notation for quotients of Γ_q functions will often be used.

$$\Gamma_q\left[\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_r \end{array}\right] \equiv \frac{\Gamma_q(a_1)\ldots\Gamma_q(a_p)}{\Gamma_q(b_1)\ldots a_q(b_r)}. \tag{15}$$

Just like in [7], the function

$$F(n)_{a,b,q} : n \to (a \oplus_q b)^n, a, b \in \mathbb{R}+, a < 1, b < 1 \tag{16}$$

is crucial for the convergence of Φ_4. This function first increases to a maximum and then decreases to 0.

We will denote some of the ensuing tables by capitals since we will later refer to them. The tables for the missing capitals can be found in the first article [7].

The following table (C) shows some function values $F(n)_{.78,.78,.9}$:

n	$F(n)_{.78,.78,.9}$
599	9.82785×10^{-57}
600	7.67976×10^{-57}
601	6.00117×10^{-57}

The following table (J) shows some function values $F(n)_{.98,.98,.89}$:

n	$F(n)_{.98,98,.89}$
599	561.34
600	551.109
601	541.062

The following table (K) shows some function values $F(n)_{.98,.98,.895}$:

n	$F(n)_{.98,98,.895}$
599	1110.86
600	1090.62
601	1070.75

2. q-Stirling proofs and numerical function values

Definition 3. The q-analogues of the two last Appell functions are

$$\Phi_3(a,a';b,b';c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}. \tag{17}$$

$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{18}$$

The following calculations are q-analogues of Appell & Kampé de Fériet [1, p. 17-18]. Consider $\Phi_3(a,a';b,b';c|q;x_1,x_2)$. The coefficient of $x_1^m x_2^n$ is equal to

$$A_{m,n} \equiv \Gamma_q \left[\begin{matrix} c, a+m, a'+n, b+m, b'+n, 1, 1 \\ a, a', b, b', c+m+n, 1+m, 1+n \end{matrix} \right]. \tag{19}$$

According to the $q-$Stirling formula, $\lim_{m,n\to\infty}$

$$A_{m,n} \sim \Gamma_q \left[\begin{array}{c} c \\ a, a', b, b' \end{array} \right] \lim_{m,n\to\infty} \{m+n\}_q^{1-c} \{m\}_q^{a+b-2} \{n\}_q^{a'+b'-2} \left(\binom{m+n}{m}_q \right)^{-1}. \tag{20}$$

The real parts of a, a', b, b', c, c' are $a_1, a_1', b_1, b_1', c_1, c_1'$ and N is a number such that

$$N > \left| \Gamma_q \left[\begin{array}{c} c \\ a, a', b, b' \end{array} \right] \right|. \tag{21}$$

For m, n big enough, we have

$$|A_{m,n} x_1^m x_2^n| < N \left(\binom{m+n}{n}_q \right)^{-1} |x_1|^m |x_2|^n \{m+n\}_q^{1-c_1} \{m\}_q^{a_1+b_1-2} \{n\}_q^{a_1'+b_1'-2}. \tag{22}$$

Also

$$|A_{m,n} x_1^m x_2^n| < N |x_1|^m |x_2|^n \{m+n\}_q^{1-c_1} \{m\}_q^{a_1+b_1-2} \{n\}_q^{a_1'+b_1'-2}. \tag{23}$$

The function Φ_3 converges for $|x_1| < 1$, $|x_2| < 1$.

The following tables show some partial sums $\Phi_3(N)$, where the summation extends to N. For every case, the function parameters are given in the table.

N	$\Phi_3(N; .7, .66; -.88, .65; .44 \vert .4; 1.0000, .9999)$
210	-2080
240	-2702.64
270	-3405.47
300	-4188.26
330	-5050.66

N	$\Phi_3(N;.7,.66;-.88,.65;.44\vert.4;1.0001,.9999)$
60	-185.883
90	-402.525
120	-702.478
150	-1085.74
180	-1552.32
210	-2102.23
240	-2735.45
270	-3452.01
300	-4251.91
330	-5135.15
360	-6101.74
390	-7151.7

In the first case, $\triangle\Phi_3 \approx -600, -700, -800, -900$. In the second case, $\triangle\Phi_3 \approx -200, -300, -400, -500, -600, \cdots, -1000$. Obviously no convergence is in sight, which vindicates the previous hypothesis.

As the following table shows, we have to decrease x to .995 to get convergence. These computations take a lot of time, five days for one function value when $N = 70$, which shows that the convergence boundary is sharp.

x	$\Phi_3(.7,.66;-.88,.65;.44\vert.4;x,x)$
.95	-23.1675
.96	-34.5432
.97	-58.62
.995	≈ -1889.1

We now leave Φ_3 permanently and turn to Φ_4. We start with a well-known formula.

Theorem 2.1. *If* $|x| < \frac{1}{4}$ *then* [4]

$$\sum_{m,n=0}^{\infty} \frac{\langle \lambda, \mu; q\rangle_{m+n} x^{m+n} q^{2\binom{n}{2}+\sigma n}}{\langle \nu, 1; q\rangle_m \langle \sigma, 1; q\rangle_n} = \sum_{n=0}^{\infty} \frac{\langle \lambda, \mu, \frac{\nu+\sigma-1}{2}, \widetilde{\frac{\nu+\sigma-1}{2}}, \frac{\nu+\sigma}{2}, \widetilde{\frac{\nu+\sigma}{2}}; q\rangle_n x^n}{\langle \nu, 1, \sigma, \nu+\sigma-1; q\rangle_n}. \tag{24}$$

A practical computation with $\lambda = \mu = \nu = \sigma = 2, q = .38$ shows that formula (24) holds for $x = .38$, when both sides have the value 682.048. We can thus count on a greater convergence area for $\Phi_4(q)$. This will be shown in a moment.

Consider $\Phi_4(a;b;c,c'|q;x_1,x_2)$. The coefficient of $x_1^m x_2^n$ is equal to

$$A_{m,n} \equiv \Gamma_q \left[\begin{array}{c} c, c', a+m+n, b+m+n, 1, 1 \\ a, b, c+m, c'+n, 1+m, 1+n \end{array} \right]. \tag{25}$$

According to the $q-$Stirling formula, $\lim_{m,n\to\infty}$

$$A_{m,n} \sim \Gamma_q \left[\begin{array}{c} c, c' \\ a, b \end{array} \right] \lim_{m,n\to\infty} \{m+n\}_q^{a+b-2} \{m\}_q^{1-c} \{n\}_q^{1-c'} \left(\binom{m+n}{m}_q \right)^2. \tag{26}$$

The real parts of a, b, c, c' are a_1, b_1, c_1, c_1' and N is a number such that

$$N > \left| \Gamma_q \left[\begin{array}{c} c, c' \\ a, b \end{array} \right] \right|. \tag{27}$$

For m, n big enough, we have

$$|A_{m,n} x_1^m x_2^n| < N \left(\binom{m+n}{n}_q \right)^2 \{m+n\}_q^{a_1+b_1-2} \{m\}_q^{1-c_1} \{n\}_q^{1-c_1'} |x_1|^m |x_2|^n. \tag{28}$$

If ϵ denotes a positive number bigger than the greatest of $1-c_1$ and $1-c_1'$ and ϵ' denotes a sufficiently large number,

$$\{m\}_q^{1-c_1} \{n\}_q^{1-c_1'} < \{m\}_q^{\epsilon} \{n\}_q^{\epsilon} < \frac{\{m+n\}_q^{2\epsilon'}}{4^{\epsilon}}. \tag{29}$$

Therefore

$$\begin{aligned} \sum_{m,n=0}^{\infty} |A_{m,n} x_1^m x_2^n| &< \frac{N}{4^{\epsilon}} \sum_{m,n=0}^{\infty} \left(\binom{m+n}{n}_q \right)^2 \{m+n\}_q^{2\epsilon'+a_1+b_1-2} |x_1|^m |x_2|^n \\ &= \frac{N}{4^{\epsilon}} \sum_{r=0}^{\infty} \{r\}_q^{2\epsilon'+a_1+b_1-2} \sum_{s=0}^{r} \left(\binom{r}{s}_q \right)^2 |x_1|^s |x_2|^{r-s} \\ &< \frac{N}{4^{\epsilon}} \sum_{r=0}^{\infty} \{r\}_q^{2\epsilon'+a_1+b_1-2} \left(\sqrt{x_1} \oplus_q \sqrt{x_2} \right)^{2r}, \end{aligned} \tag{30}$$

where $r = m+n$. In the last inequality we have used the fact that

$$\sum_{s=0}^{r} \left(\binom{r}{s}_q \right)^2 |x_1|^s |x_2|^{r-s} < \sum_{k=0}^{2r} \binom{2r}{k}_q \left(\sqrt{x_1} \right)^k \left(\sqrt{x_2} \right)^{2r-k}. \tag{31}$$

The series converges for $\left(\sqrt{x_1} \oplus_q \sqrt{x_2}\right)^{2r} < 1$ and Φ_4 converges in the same region.

For q=1 the convergence region for Φ_4 is the area inside the curve $\sqrt{x_1}+\sqrt{x_2 < 1}$. In our case, the convergence region is q-dependent and the farer away q is below 1 $(0 < q < 1)$, the greater convergence region.

For the ensuing computations we give some examples of values of squareroots in order to compare with the values of $F(n)$ from the first section and from the first paper.

x	$\sqrt{x}$	x	$\sqrt{x}$	x	$\sqrt{x}$
.53	0.728011	.54	0.734847	.55	0.74162
.56	0.748331	.57	0.754983	.58	0.761577
.59	0.768115	.6	0.774597	.61	0.781025

We have collected some approximate function values in the following tables to show the extended convergence region for $\Phi_4(.7; .66; .65, .44|q; x, x)$. Compare with table C.

q	x	$\Phi_4(.7;.66;.65,.44\|q;x,x)$
.90	.53	3207.81
.90	.54	4681.62
.90	.55	6876.04
.90	.56	10162.2
.90	.57	15111.3
.90	.58	22607.4
.90	.59	34026.2
.90	.60	51519.7
.90	.61	78473.6
.90	.62	120244.
.90	.63	185355.
.90	.64	287447.
.90	.65	448485.
.91	.54	10515.2
.91	.55	16110.8
.91	.56	24854.3
.91	.57	38603.1
.91	.58	60359.7
.91	.59	95005.6
.91	.60	150526
.91	.61	240062
.92	.53	17975.2
.92	.54	28809.3
.92	.55	46535
.92	.56	75745.4
.92	.57	124226
.92	.58	205262
.92	.59	341675
.92	.60	572937
.92	.61	967781

.93	.51	21401.8
.93	.52	36008.3
.93	.53	61147.4
.93	.54	104781
.93	.55	181147
.93	.56	315906
.93	.57	555652
.93	.58	985648
.94	.41	418.156
.94	.42	672.457
.94	.43	1098.85
.94	.44	1823.06
.94	.45	3068.54
.94	.46	5236.51
.94	.47	9054.63
.94	.48	15855.7
.94	.49	28104.9
.94	.50	50405.1
.95	.36	86.1905
.95	.37	136.131
.95	.38	220.557
.95	.39	366.038
.95	.40	621.392
.95	.41	1077.63
.95	.42	1906.85
.95	.43	3439
.95	.44	6315.26
.95	.45	11798

.96	.33	41.2668
.96	.34	65.683
.96	.35	108.582
.96	.36	186.087
.96	.37	329.939
.96	.38	603.97
.96	.39	1139.2
.96	.40	2209.97
.96	.41	4402.02
.96	.42	8989.66

.97	.31	29.889
.97	.32	49.7418
.97	.33	87.6128
.97	.34	162.978
.97	.35	319.256
.97	.36	656.444
.97	.37	1412.24
.97	.38	3169.32
.97	.39	7399.1
.97	.40	17926.
.97	.41	44970.8
.97	.42	116594.
.97	.43	311860.
.97	.44	859224.

q	$\Phi_4(.7;.66;.65,.44\|q;.4356,.4356)$
.701	167.98
.81	2198.98

We now consider $\Phi_4(.7;.66;.65,.44|q;x,x)$ as a function of q for fixed x.

q	$\Phi_4(.7;.66;.65,.44\|q;.45,.45)$	q	$\Phi_4(.7;.66;.65,.44\|q;.45,.45)$
.41	12.3492	.42	13.0127
.43	13.7373	.44	14.5305
.45	15.4009	.46	16.3583
.47	17.4144	.48	18.5823
.49	19.8778		

q	$\Phi_4(80;.7;.66;.65,.44\|q;.45,.45)$	q	$\Phi_4(80;.7;.66;.65,.44\|q;.45,.45)$
.51	22.9276	.52	24.7286
.53	26.7522	.54	29.0342
.55	31.6174	.56	34.5534
.57	37.9046	.58	41.7467
.59	46.1725	.6	51.2965

q	$\Phi_4(.7;.66;.65,.44\|q;.45,.45)$	q	$\Phi_4(.7;.66;.65,.44\|q;.45,.45)$
.61	57.26	.62	64.2399
.63	72.4584	.64	82.1968
.65	93.8148	.66	107.776
.67	124.683	.68	145.329
.69	170.766		

q	$\Phi_4(.7;.66;.65,.44\|q;.45,.45)$	q	$\Phi_4(.7;.66;.65,.44\|q;.45,.45)$
.7	202.409	.71	242.183
.72	292.742	.73	357.8
.74	442.633	.75	554.87
.76	705.745	.77	912.142
.78	1200.01	.79	1610.23

For the last value in the next table one should refer to table B from [7].

q	$\Phi_4(.7;.66;.65,.44\|q;.5,.5)$
.886	917430.
.96	3.38223×10^{12}

Compare with table C from [7]:

q	$\Phi_4(.7;.66;.65,.44\|q;.6,.6)$
.844	789396.
.845	848891.
.846	913577.
.847	983965.

Compare with table D from [7]:

q	$\Phi_4(.7;.66;.65,.44\|q;.64,.64)$
.96	917430.

q	x	$\Phi_4(.7;.66;.65,.44\|q;x,x)$
.81	.69	973940.
.82	.664	920101.

For the next three tables, one should compare with tables G_1, G_2 from [7]:

q	$\Phi_4(.7;.66;.65,.44\vert q;.96,.96)$
.80	9.67×10^9
.85	4.875×10^{12}
.88	1.935×10^{15}

q	$\Phi_4(.7;.66;.65,.44\vert q;.97,.97)$
.81	6.07734×10^{10}
.86	7.2639×10^{13}

q	$\Phi_4(.7;.66;.65,.44\vert q;(.985)^2,(.985)^2)$
.895	7.04×10^{13}

Compare with table H from [7]:

q	$\Phi_4(.7;.66;.65,.44\vert q;.98,.98)$
.82	5.602×10^{11}
.84	8.2267×10^{12}
.87	1.91501×10^{15}

In the last table we show that the convergence area is not changed very much if we change one of the parameters.

q	x	$\Phi_4(.7;.66;.65,.49\vert q;x,x)$
.90	.54	3849.33
.90	.55	5647.26
.90	.56	8337.29
.90	.57	12385.2
.90	.58	18511.4
.90	.59	27836.1
.90	.60	42110.8
.90	.61	64089.4

3. Conclusion

We have found that the convergence regions for the third and fourth q-Appell functions follow the same pattern as for the first and second q-Appell functions from [7]. The result is an increment of the convergence region for Φ_4 and status quo for Φ_3. Corresponding results for q-Lauricella functions will follow in ensuing articles, see [8]. Part of this paper was presented at the ICNAAM meeting 2010.

References

[1] P.Appell and J.Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques.* Paris 1926.

[2] O. Daalhuis, Asymptotic expansions for q-gamma, q-exponential, and q-Bessel functions. *J. Math. Anal. Appl.* **186** (1994), no. 3, 896–913.

[3] T. Ernst, A method for q-calculus. *J. nonlinear Math. Physics* **10** No.4 (2003) 487-525.

[4] T. Ernst, Some results for q-functions of many variables. *Rendiconti di Padova,***112** (2004), 199-235.

[5] T. Ernst, Some new formulas involving Γ_q functions. *Rendiconti di Padova* **118** (2007), 159–188.

[6] T. Ernst, q-analogues of general reduction formulas by Buschman and Srivastava and an important q-operator reminding of MacRobert. *Demonstratio Mathematica* **44** (2011), 285-296

[7] T. Ernst, Convergence aspects of for q–Appell functions I. Submitted.

[8] Convergence aspects for q–Lauricella functions I. Adv. Studies Contemp. Math. **22** (1), 2012

[9] T. Ernst, *A comprehensive treatment of q-calculus*, Birkhäuser 2012

[10] Exton H., *Multiple hypergeometric functions and applications.* Mathematics & its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York-London-Sydney, 1976.

[11] G. Gasper and M. Rahman, *Basic hypergeometric series,* Cambridge, 1990.

[12] F.H.Jackson, On basic double hypergeometric functions. *Quart. J. Math.*, Oxford Ser. **13** (1942), 69–82.

[13] D.S.Moak, The q-analogue of Stirling's formula. *Rocky Mountain J. Math.* **14** (1984), no. 2, 403–413.

[14] W.Rudin, *Real and complex analysis.* McGraw-Hill 1987.

[15] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian hypergeometric series.* Ellis Horwood, New York, 1985.

Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden

E-mail address: thomas@math.uu.se