q-COMPLEX NUMBERS, A NATURAL CONSEQUENCE OF UMBRAL CALCULUS

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1. Introduction

We are going to treat a kind of complex numbers in this paper. Since Nordic mathematics is not too famous, we start with a short account of the first occurrence of the wellknown complex plane. We then proceed with basic information about q-calculus. In chapter two we introduce the q-umbral calculus of the author. In chapter three we present the q-complex numbers in umbral form, the main content of this paper. In chapter four we briefly discuss the history behind this new concept. A motivation and longer introduction to this history is given in [12].

Caspar Wessel (1745–1818), the sixth of fourteen children, was born in Norway, but moved to Copenhagen in 1763 to study law. Wessel started to work as a surveyor 1764; this work culminated in the first really accurate map of Denmark 1796. He possessed a lot of theoretical knowledge of algebra, trigonometry and mathematical geometry, and as far as the last point is concerned, he has come up with some new and beautiful solutions to the most difficult problems in geographical surveying. These he fully explained in a report from 1787. This report already contains Wessel's brilliant geometric interpretation of complex numbers. Wessel's paper was presented to the Danish Academy 1797 and appeared in print 1799. It is also claimed that Wessel was the first person to add vectors; his presentation was superior and more modern in spirit compared to Argand.

Heine [18] tried to combine certain q-hypergeometric series with elliptic functions. This so-called Heine q-umbral calculus reached its peak in the thesis by Smith [30] 1911, supervised by Alfred Pringsheim.

We will now describe the q–umbral method invented by the author [4]– [10], which also involves the Nalli–Ward-Alsalam (NWA) q–addition and the Jackson–Hahn–Cigler (JHC) q–addition. This method

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is a mixture of Heine 1846 and Gasper-Rahman [13]. The advantages of this method have been summarized in [7, p. 495].

Definition 1. The power function is defined by $q^a \equiv e^{alog(q)}$. We always use the principal branch of the logarithm. The variables

$$a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$$

denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The q–analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{1}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n}\{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}, \tag{2}$$

Definition 2. Let the q–shifted factorial (compare [14, p.38]) be defined by

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots, \end{cases} \tag{3}$$

The Watson notation [13] will also be used

$$(a;q)_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\ldots \end{cases} \tag{4}$$

Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_{n-k}},\ k=0,1,\ldots,n. \tag{5}$$

The q-multinomial coefficient is defined by

$$\binom{n}{\vec{k}}_q \equiv \frac{\langle 1;q\rangle_n}{\prod_{i=1}^{\infty}\langle 1;q\rangle_{k_i}},\ \sum_{i=1}^{\infty} k_i = n. \tag{6}$$

The following notation will sometimes be used:

$$P_{n,q}(x,y) \equiv x^n(-\frac{y}{x};q)_n,\ n=0,1,2,\ldots. \tag{7}$$

Definition 3. Let a and b be any elements with commutative multiplication. Then the NWA q-addition, compare [1, p. 240], [23, p. 345], [33, p. 256] is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \; n = 0, 1, 2, \ldots. \tag{8}$$

Furthermore,we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k}, \; n = 0, 1, 2, \ldots. \tag{9}$$

There is a q-addition dual to the NWA, which will be presented here for reasons to be given shortly. The following polynomial in 3 variables x, y, q originates from Gauss.

Definition 4. The JHC q-addition, compare [3, p. 91], [17, p. 362], [21, p. 78] is the function

$$(x \boxplus_q y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} y^k x^{n-k} = P_{n,q}(x, y), \; n = 0, 1, 2, \ldots. \tag{10}$$

$$(x \boxminus_q y)^n \equiv P_{n,q}(x, -y), \; n = 0, 1, 2, \ldots. \tag{11}$$

2. The q-umbral calculus

We first give an improved and extended version of the q-umbral calculus from [9] and [2]. This umbral calculus has a close connection to the work of Nörlund [25] on Bernoulli polynomials. For the rest of the paper, we assume that $0 < |q| < 1$.

Definition 5. The identity operator and the forward operator are denoted I and E.

Let $\{\theta_i\}_0^\infty$ and $\{\phi_i\}_0^\infty$ denote arbitrary secuences. The Carlitz–Gould [15] q-difference operator is defined by

$$\triangle_{\mathrm{CG},q}\theta_0 \equiv (E - I)\theta_0, \; \triangle_{\mathrm{CG},q}^{l+1}\theta_0 \equiv \triangle_{\mathrm{CG},q}^{l} E\theta_0 - q^l \triangle_{\mathrm{CG},q}^{l}\theta_0. \tag{12}$$

Or equivalently,

$$\triangle_{\mathrm{CG},q} \equiv E \boxminus_q I. \tag{13}$$

The last formula requires knowledge of the umbral calculus to be defined shortly. If the identity operator, the forward operator, or the Carlitz–Gould operator operates on m, we denote them I_m, E_m, or $\triangle_{\mathrm{CG},m,q}$.

This implies

$$\triangle_{\mathrm{CG},q}^{l}\theta_0=\prod_{s=0}^{l-1}(E-q^s)\theta_0=\sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}_q q^{\binom{l-k}{2}}E^k\theta_0. \tag{14}$$

$$E^l\theta_0=\sum_{k=0}^{l}\binom{l}{k}_q \triangle_{\mathrm{CG}q}^{k}\theta_0. \tag{15}$$

$$\triangle_{\mathrm{CG},q}^{l}(\theta_0\phi_0)=\sum_{k=0}^{l}\binom{l}{k}_q \triangle_{\mathrm{CG},q}^{k}\theta_0\triangle_{\mathrm{CG},q}^{l-k}E^k\phi_0. \tag{16}$$

The following formula in the spirit of Sears [26, p. 15] and [29, p. 159] gives the q-difference for quotients of q-shifted factorials.

Theorem 2.1.

$$\triangle_{\mathrm{CG},m,q}^{k}\frac{\langle a_l;q\rangle_m}{\langle b_l;q\rangle_m}=(-1)^k\frac{\langle a_l;q\rangle_m\langle b_l-a_l;q\rangle_k}{\langle b_l;q\rangle_{m+k}}q^{k(a+m)+\binom{k}{2}} \tag{17}$$

We are now going to present a q-umbral calculus, which is somewhat influenced by [28, p. 696]. We have however made certain changes from the original Rota-Taylor paper. Before Rota's paper, E.T. Bell [2] wrote a conceptual introduction to umbral calculus, with vector functions, which we will follow slightly.

Definition 6. We consider functions of n variables $x_1,\ldots,x_n$, and m parameters $y_1,\ldots,y_m$. A q-umbral calculus contains a set A, called the alphabet, with elements called letters or umbrae. The letter α is connected to the variable x_i if e.g. $\alpha=x_i$. The letter β is connected to the parameter $y_i=k$ if either

(1) $\beta=I_k$
(2) $\beta=E_k$
(3) $\beta=\triangle_{\mathrm{CG},k,q}$

Assume that α, β, are distinct umbrae, then a new umbra is obtained by $\alpha*\beta$, where $*$ is $\oplus_q$, $\boxplus_q$, $\ominus_q$, $\boxminus_q$, or any general q-addition.

There is a certain linear functional $eval$, $\mathbb{C}[[x]]\times A\to\mathbb{C}$, called the evaluation. In the following, an arbitrary $f\in\mathbb{C}[[x]]$ will be used.

Two umbrae α and β are said to be equivalent, denoted $\alpha\sim\beta$ if $eval(f,\alpha)=eval(f,\beta)$. The set of equivalent umbrae form an equivalence class.

There is a distinguished element θ of the alphabet called the zero, such that

$$x\boxminus_q x\sim\theta. \tag{18}$$

We also usually have

$$eval(f,\theta)=1. \tag{19}$$

Elements α and $\beta \in A$ are said to be inverse to each other if $\alpha \boxplus_q \beta \sim \theta$.

We can also define powers and products of letters; this should however be done in a final stage, just before applying the linear functional *eval*. Powers of letters are defined as α^j etc. Two different letters are called disjoint if they are connected to different variables or parameters. If $\alpha,\ \beta,\ \dots,\ \gamma$ are disjoint umbrae, and $i, j,\ \dots,\ k$ positive integers, the expression

$$eval(f, \alpha^i \beta^j \dots \gamma^k) \tag{20}$$

makes sense.

There is a Ward number $\bar{k}_q$

$$\bar{k}_q \sim 1 \oplus_q 1 \oplus_q \dots \oplus_q 1, \tag{21}$$

where the number of 1 in the RHS is k.

There is a Jackson number $\tilde{k}_q$

$$\tilde{k}_q \sim 1 \boxplus_q 1 \boxplus_q \dots \boxplus_q 1, \tag{22}$$

where the number of 1 in the RHS is k.

There is a multiplication with Ward- and Jackson numbers in the following sense: Assume that $a \in \mathbb{C}$ or $a \in \mathbb{C}[D_q]$. Then we define

$$a\bar{k}_q \sim a \oplus_q a \oplus_q \dots \oplus_q a, \tag{23}$$

where the number of a in the RHS is k. In the same way,

$$a\tilde{k}_q \sim a \boxplus_q a \boxplus_q \dots \boxplus_q a, \tag{24}$$

where the number of a in the RHS is k.

If $\alpha_1, \dots, \alpha_k\ \in A,\ \alpha \sim \alpha_i,\ i = 1, \dots, n$ then $\alpha_1 \oplus_q \dots \oplus_q \alpha_k \sim \bar{k}_q \alpha$.

Remark 1. In the original paper [28, p. 696], certain powers of umbrae were computed without a definition of multiplication in the alphabet. Influenced by this another condition in the definition of q-umbral calculus was given in the previous paper [9]. This extra condition, which was never used has now been changed.

Example 1. Jackson [20, (8), p.146]. If $\{\phi_j\}_{j=0}^{\infty}$ is an arbitrary sequence,

$$\sum_{k=0}^{\infty} \frac{x^k \phi_k}{\{k\}_q!} = E_q(x) \sum_{k=0}^{\infty} \frac{x^k \triangle_{\mathrm{CG},q}^k}{\{k\}_q!} \phi_0. \tag{25}$$

This formula can be written in umbral notation as

$$E_q(xE_j) = E_q(x(I_j \oplus_q E_j \boxminus_q I_j)). \tag{26}$$

Definition 7. We use n variables $x_1, \dots, x_n \in \mathbb{C}$. The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1, \dots, m_n$ running over all non-negative integer values. In this connection we put $|m| \equiv \sum_{j=1}^{n} m_j$.

If $\vec{m}$ and $\vec{k}$ are two arbitrary vectors of positive integers with n elements, their q-binomial coefficient is defined as

$$\binom{\vec{m}}{\vec{k}}_{\vec{q}} \equiv \prod_{j=1}^{n} \binom{m_j}{k_j}_{q_j}. \tag{27}$$

In the same way we define vector versions of powers, q-shifted factorials, etc.

$$\vec{x}^{\vec{\alpha}} \equiv \prod_{j=1}^{n} x_j^{\alpha_j}, \tag{28}$$

$$\frac{1}{(\vec{x};\vec{q})_{\vec{\beta}}} \equiv \prod_{j=1}^{n} \frac{1}{(x_j;q_j)_{\beta_j}}, \tag{29}$$

$$\langle\vec{\alpha};\vec{q}\rangle_{\vec{k}} \equiv \prod_{j=1}^{n} \langle\alpha_j;q_j\rangle_{k_j}, \tag{30}$$

$$\{l\}_{\vec{q}}! \equiv \prod_{j=1}^{n} \{l_j\}_{q_j}!, \tag{31}$$

$$\vec{q}^{\,\binom{\vec{k}}{2}} \equiv \prod_{j=1}^{n} q_j^{\binom{k_j}{2}}, \tag{32}$$

$$(-1)^{\vec{k}} \equiv (-1)^{|k|}. \tag{33}$$

3. q-complex numbers

Throughout the variable z will denote certain q-complex numbers, the main theme to be introduced shortly.

We are going to introduce two kinds of q-complex numbers.

Definition 8. We define the q-complex numbers $\mathbb{C}^{\star}_{\oplus_q}$ as the set $\{z = x \oplus_q iy\}$, where x, y each belong to an infinte set generated by real letters together with NWA.

Definition 9. We define the q-complex numbers $\mathbb{C}_{\oplus_q}$ as the set $\{z = x \oplus_q iy\}, x, y \in \mathbb{R}$.

x and y are called the real and imaginary parts of z, denoted Re z and Im z.

Theorem 3.1.
$$\mathbb{C}_{\oplus_q} \subset \mathbb{C}^{\star}_{\oplus_q} \subset A. \tag{34}$$

Definition 10. For each $z \in \mathbb{C}^{\star}_{\oplus_q}$ or $z \in \mathbb{C}_{\oplus_q}$, $\alpha \in \mathbb{R}$, we define a scalar multiplication
$$\alpha(z) \equiv \alpha z. \tag{35}$$

Theorem 3.2. *In the following we will use the notation*
$$z_j \equiv x_j \oplus_q iy_j, \;\; z_j \in \mathbb{C}_{\oplus_q}. \tag{36}$$

We define an addition $\oplus_q : \mathbb{C}_{\oplus_q} \times \mathbb{C}_{\oplus_q} \mapsto \mathbb{C}^{\star}_{\oplus_q}$
$$z_1 \oplus_q z_2 \sim (x_1 \oplus_q x_2) \oplus_q i(y_1 \oplus_q y_2). \tag{37}$$
A subtraction $\ominus_q : \mathbb{C}_{\oplus_q} \times \mathbb{C}_{\oplus_q} \mapsto \mathbb{C}^{\star}_{\oplus_q}$
$$z_1 \ominus_q z_2 \sim (x_1 \ominus_q x_2) \oplus_q i(y_1 \ominus_q (y_2)). \tag{38}$$
A product $\odot_q : \mathbb{C}_{\oplus_q} \times \mathbb{C}_{\oplus_q} \mapsto \mathbb{C}_{\oplus_q}$
$$z_1 \odot_q z_2 \sim (x_1x_2 - y_1y_2) \oplus_q i(x_1y_2 + y_1x_2). \tag{39}$$
A quotient $\ominus_q : \mathbb{C}_{\oplus_q} \times \mathbb{C}_{\oplus_q} \mapsto \mathbb{C}_{\oplus_q}$
$$z_1 \ominus_q z_2 \sim \frac{1}{\sqrt{x_2^2 + y_2^2}}\,(\,x_1x_2 + y_1y_2) \oplus_q i(x_2y_1 - x_1y_2). \tag{40}$$

There exists $0 \in \mathbb{C}_{\oplus_q}$*, called the zero, such that*
$$z \oplus_q 0 \sim z. \tag{41}$$
This corresponds to θ *in the alphabet. There is no additive inverse, because the NWA q-addition forms a monoid. However, if we may also use* $\boxplus_q$*, we can come back to the first value after one q-addition and one q-subtraction.*

There exists $1 \in \mathbb{C}_{\oplus_q}$*, called the unit, such that*
$$z \odot_q 1 \sim 1 \odot_q z \sim z. \tag{42}$$
The product $\odot_q$ *is commutative.*

Definition 11. The absolute value of $z \in \mathbb{C}_{\oplus_q}$ is given by
$$|z| \equiv \sqrt{x^2 + y^2}. \tag{43}$$
The absolute value of $z_1 \ominus_q z_2 \in \mathbb{C}^{\star}_{\oplus_q}$ is given by
$$|z_1 \ominus_q z_2| \equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{44}$$

The conjugate of $z \in \mathbb{C}_{\oplus_q}$ is given by
$$\overline{z} \sim x \ominus_q iy. \tag{45}$$

The conjugate of $z_1 \oplus_q \ldots \oplus_q z_m \in \mathbb{C}^{\star}_{\oplus_q}$ is given by

$$\overline{(z_1 \oplus_q \ldots \oplus_q z_m)} \sim \overline{z_1} \oplus_q \ldots \oplus_q \overline{z_m}. \tag{46}$$

Theorem 3.3. *Some rules for the conjugate:*

$$z \odot_q \overline{z} \sim x^2 + y^2. \tag{47}$$

$$|z| = |\overline{z}|. \tag{48}$$

$$\overline{(z_1 \odot_q z_2)} \sim \overline{z_1} \odot_q \overline{z_2}. \tag{49}$$

Definition 12. Limits and q-continuity for q-complex numbers. Assume that we have chosen an element $f \in \mathbb{C}[[z]]$, and a q-complex number $z_0 \in \mathbb{C}_{\oplus_q}$. Then limits of q-complex numbers are defined as follows:

$$\lim_{z \to z_0} f(z) = \alpha \tag{50}$$

means that for every positive ϵ there is a positive δ such that

$$0 < |z \ominus_q z_0| < \delta \text{ implies } |f(z) - \alpha| < \epsilon. \tag{51}$$

The function $f \in \mathbb{C}[[z]]$ is called q-continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{52}$$

Definition 13. If $f(z) \in \mathbb{C}[[z]]$, $z \in \mathbb{C}_{\oplus_q}$ we define the complex q-derivative as

$$D_{\oplus_q} f(z) \equiv D_{\oplus_{q,z} f(z)} \equiv \lim_{\delta z \to 0} \frac{f(z \oplus_q \delta z) - f(z)}{(\delta z)^1}. \tag{53}$$

The following system of equations (for $q = 1$) first appeared in the works of D'Alembert in 1752 about wave equations. Later, in 1777, Euler connected this system to the analytic functions.

Theorem 3.4. *Assume that $f(z) \in \mathbb{C}[[z]]$ can be divided in real and imaginary parts as $f(z) = u(x,y) + iv(x,y), z = x \oplus_q iy \in \mathbb{C}_{\oplus_q}$. Then the q-Cauchy-Riemann equations obtain.*

$$D_{q,x} u(x,y) = D_{q,y} v(x,y), \tag{54}$$

$$D_{q,y} u(x,y) = -D_{q,x} v(x,y). \tag{55}$$

Theorem 3.5. *Let $f(z) \in \mathbb{C}[[z]]$, then*

$$D_{\oplus_q} f(z) = D_{q,x} u(x,y) + i D_{q,x} v(x,y). \tag{56}$$

Corollary 3.6.

$$D_{\oplus_q} z^k = \{k\}_q z^{k-1}. \tag{57}$$

Definition 14. The complex and conjugated complex q-difference operators are given by

$$D_{q,z}f(z) \equiv \frac{D_{q,x} - iD_{q,y}}{2} f(z), \tag{58}$$

$$D_{q,\overline{z}}f(z) \equiv \frac{D_{q,x} + iD_{q,y}}{2} f(z). \tag{59}$$

Definition 15. We can extend these definitions to complex functions of n variables in the following way, a q-analogue of [27, p. 5, 1.8].

$$D^{\vec{l}}_{\vec{q},\vec{x}} F(\vec{x},\vec{q}) \equiv \prod_{j=1}^{n} (D^{l_j}_{q_j,x_j}) F(\vec{x},\vec{q}), \tag{60}$$

$$D^{\overline{lk}}_{\vec{q},\vec{x}} F(\vec{x},\vec{q}) \equiv \prod_{j=1}^{n} (D^{l_j}_{q_j,z_j} D_{q_j,\overline{z_j}^{k_j}}) F(\vec{x},\vec{q}),\ \vec{l},\vec{k} \in \mathbb{N}^n. \tag{61}$$

Definition 16. The function $f(z)$ is called q-holomorphic if and only if

$$D_{q,\overline{z}}f(z) = 0. \tag{62}$$

Corollary 3.7. *This is easily seen to be equivalent to the statement* $f \in \mathbb{C}[[z]]$.

Corollary 3.8. *The function $f(z)$ is q-holomorphic if and only if*

$$D_{\oplus_q} f(z) = D_{q,x} f(x \oplus_q iy). \tag{63}$$

Proof. Use the q-Cauchy-Riemann equations. □

Theorem 3.9. *All q-holomorphic functions are q-continuous.*

Definition 17. The function $f(z)$ of n variables is called q-holomorphic if and only if

$$D_{q,\overline{z_j}} f(z) = 0,\ 1 \le j \le n. \tag{64}$$

We denote the class of these functions, the formal power series, $F(\vec{x}) \in (\mathbb{C}(\vec{q}))[[\vec{x}]]$ by $\vec{\mathrm{H}}_{\vec{q},n}$.

We have the following q-analogue of the Laplace equation.

Definition 18. A function $u(x,y)$ of two variables is called q-harmonic if

$$D^2_{q,x}u + D^2_{q,y}u = 0. \tag{65}$$

Theorem 3.10. *If $f(z) = u(x,y) + iv(x,y)$ is q-holomorphic, the real and imaginary parts u and v are q-harmonic.*

By the q-binomial theorem, we can define inverse q-shifted factorials of q-complex numbers in the following formal way.

Definition 19. Let $z \in \mathbb{C}_{\oplus_q},\ |z| < 1, a \in \mathbb{C}$. Then the inverse q-shifted factorial is defined by

$$\frac{1}{(z;q)_a} \cong \sum_{k=0}^{\infty} \frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k} z^k. \tag{66}$$

The symbol $\cong$ denotes that the equality is purely formal.

Theorem 3.11.

$$D_{\oplus_q} \frac{1}{(z;q)_a} = \{a\}_q \frac{1}{(z;q)_{a+1}}. \tag{67}$$

Definition 20. We define meromorphic continuation in the following way: Let z_1 and z_2 be two arbitrary q-complex, or complex numbers, and let $F(z)$ be (q-)holomorphic. Further assume that $F(z)$ has a convergent power series expansion around zero, (with convergence radius $r > |z_1|$). Then there are at least two meromorphic continuations to the umbrae $z_1 \oplus_q z_2$ and $z_1 \boxplus_q z_2$.

$$F(z_1 \oplus_q z_2) = \sum_{n=0}^{\infty} \frac{z_2^n}{\{n\}_q!} D_q^n F(z_1). \tag{68}$$

$$F(z_1 \boxplus_q z_2) = \sum_{n=0}^{\infty} \frac{z_2^n}{\{n\}_q!} q^{\binom{n}{2}} D_q^n F(z_1). \tag{69}$$

The absolute values of z_1 and z_2 should be moderate for proper convergence.

Theorem 3.12. *If we decide to use only $\oplus_q$, every q-holomorphic function, which is defined by a formal power series in a limited region in $\mathbb{C}$, can be meromorphically continued in $\mathbb{C}$.*

A generalization of the Taylor expansions (68) and (69) to functions of n q-complex, or complex numbers in the spirit of [32, p.39] *can also be done. When we have started to talk about functions of n variables, we can can continue with a few standard theorems.*

Definition 21. A q-analogue of Range [27, p. 19].

Let $E \subset \mathbb{C}^n$ and consider a map $F : E \mapsto \mathbb{C}^m$. Write $F = (f_1, \ldots, f_m)$ and $f_k = u_k + iv_k$, with u_k, v_k real valued functions on E. Then we can view $F = (u_1, v_1 \ldots, u_m, v_m)$ as a map from $E \subset \mathbb{R}^{2n}$ into $\mathbb{R}^{2m}$. The Jacobian matrix is given by

$$J_{q,\mathbb{R}}(F)(a) \equiv \begin{vmatrix} D_{q,x_1}u_1 & D_{q,y_1}u_1 & \cdots & D_{q,y_n}u_1 \\ D_{q,x_1}v_1 & D_{q,y_1}v_1 & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ D_{q,x_1}v_m & D_{q,y_1}v_m & \cdots & D_{q,y_n}v_m \end{vmatrix} \tag{70}$$

computed at $a \in E$.

The map F is called q-holomorphic if its components are q-holomorphic. In this case we have

Theorem 3.13. *The q-difference of the map $F : E \mapsto \mathbb{C}^m$ has the complex matrix representation*

$$D_q F(a) = \begin{vmatrix} D_{q,z_1} f_1 & \cdots & D_{q,z_n} f_1 \\ \vdots & \cdots & \vdots \\ D_{q,z_1} f_m & \cdots & D_{q,z_n} f_m \end{vmatrix} \tag{71}$$

Lemma 3.14. *det* $J_{q,\mathbb{R}}(F)(z) = |detD_q F(z)|^2 \geq 0,\ z \in E,\ E \subset \mathbb{C}^n.$

We can also q-deform the rational numbers [31, p. 170] *in a natural way.*

Definition 22. The q-rational numbers $\mathbb{Q}_{\oplus_q}$ are defined as all q-complex numbers $\{z = x \oplus_q iy\} x, y \in \mathbb{Q}$.

Theorem 3.15.

$$\mathbb{Q}_{\oplus_q} \subset \mathbb{C}_{\oplus_q}. \tag{72}$$

Definition 23. We define the relation $\simeq_f$ (the subscript is for fraction) on $\mathbb{Q}_{\oplus_q}$, such that if

$$\{z_i = x_i \oplus_q iy_i\}_{i=1}^{\infty},\ z_i \in \mathbb{Q}_{\oplus_q}, x_i = \frac{m_i}{n_i},\ y_i = \frac{k_i}{l_i}, \tag{73}$$

$$z_1 \simeq_f z_2 \Leftrightarrow m_1 n_2 = m_2 n_1,\ k_1 l_2 = k_2 l_1 \tag{74}$$

Theorem 3.16. $\simeq_f$ *is an equivalence relation on* $\mathbb{Q}_{\oplus_q}$.

4. Discussion

The q-complex numbers form a logical continuation of several mathematical disciplines, which we will briefly summarize. The complex numbers introduced by Rafael Bombelli and Viète in different notation, were used by Leibniz and Johann Bernoulli to develop the calculus. Hindenburg started the first modern school of combinatorics with the intention that this subject should occupy a major position in mathematics. He invented an unusual notation for powers of sums of many variables, but still had many pupils who developed combinatorics in different directions. Examples are Rothe who found the q-binomial theorem; Kramp who invented the factoriellen, a side-track of the Γ*-function; Pfaff and Gauss who developed the hypergeometric function, the corner-stone of special functions; Christoph Gudermann, the great master of elliptic functions, and the teacher of Weierstrass, who liked to develop functions in series. One of the aims of the present paper is to follow this*

Hindenburg path. We give one example. According to Netto [24]*, the so-called multinomial expansion theorem was first mentioned in a letter 1695 from Leibniz to Johann Bernoulli, who proved it. The multinomial expansion theorem was a central formula in the Hindenburg combinatorial school* [22, p. 195].

A natural q-analogue is given by:

If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, it's k'th NWA-power is given by

$$(\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q. \quad (75)$$

The umbral calculus invented by Lucas and Blissard is necessary to describe the interaction between q-complex numbers. The $\dot{+}$ operator invented by E.T. Bell [2, p. 723] *is replaced by the Nalli q-addition; the E.T. Bell umbral calculus for functions of many variables is replaced here by a notation in the spirit of Hörmander. Although Bell's graduate student Morgan Ward* [33] *reinvented the NWA q-addition, Bell himself never used q-calculus. Henry Gould, a recently retired prof. in Morgantown, West Virginia has developed a calculus in the same vein as the author. Gould's q-difference operator is found in chapter 2, and his book about combinatorial identities* [16]*, in the spirit of Hindenburg, is known in Norway.*

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