q-BERNOULLI AND q-STIRLING NUMBERS, AN UMBRAL APPROACH

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1. Introduction

The aim of this paper is to describe how different q-difference operators combine with q-Bernoulli, q-Euler and q-Stirling numbers to form various q-formulas.

The Bernoulli numbers were first used by Jacob Bernoulli (1654-1705) [13], who calculated the sum

$$s_m(n) \equiv \sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} n^{m+1-k} B_k. \tag{1}$$

In 1738 Euler used the generating functions to study the Bernoulli polynomials. The Bernoulli polynomials were also studied by J.-L. Raabe (1801-1859) [106] and Schlömilch. Raabe found two important formulas for these polynomials.

James Stirling (1692–1770), who was born in Scotland, was a contemporary of Euler, who studied in Glasgow and Oxford. Stirling expressed Maclaurins formula in a different form using what is now called Stirling's numbers of the second kind [53, p. 102].

A.T. Vandermonde (1735–1796) is best known for his determinant and for the Vandermonde theorem for hypergeometric series [137]. Vandermonde also introduced the following notation in 1772 [137].

Let $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ be the falling factorial. Stirling numbers of the first kind are the coefficients in the expansion $(x)_n = \sum_{k=0}^{n} s(n,k)(x)^k$. The Stirling numbers of the second kind are given by $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$.

In combinatorics, unsigned Stirling numbers of the first kind $|s(n,k)|$ count the number of permutations of n elements with k disjoint cycles.

Tables of the first $S(n,k)$ were given by De Morgan [35, p. 253] and by Grünert [63], [64, p. 279]. These tables were subsequently extended by Cayley [23].

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The sum $B_n = \sum_{k=1}^{n} S(n,k)$ is the nth Bell number [58, p 456, 4.12], with applications in partition theory [7].

Gould [57] remarks that many sums involving binomial coefficients greatly benefit from the use of Bernoulli- Euler- or Stirling numbers. Bernoulli- and Stirling numbers have wideranging applications in computer technology [61] and in numerical analysis [46]. One reason is that computers use difference operators rather than derivatives and these numbers are used in the transformation process.

Bernoulli- and Stirling numbers also have applications in statistics as was shown in the monograph by Jordan [80, p, 14].

The Bernoulli- and Stirling numbers are intimately connected by a well-known formula; they complement each other.

We will now describe the q–umbral method invented by the author [37]– [42], which also involves the Nalli–Ward-Alsalam q–addition and the Jackson–Hahn–Cigler q–addition. This method is a mixture of Heine 1846 [68] and Gasper-Rahman [47]. The advantages of this method have been summarized in [40, p. 495].

Definition 1. The power function is defined by $q^a \equiv e^{alog(q)}$. We always use the principal branch of the logarithm. The variables

$$a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$$

denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The q–analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{2}$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}, \tag{3}$$

Definition 2. Let the q–shifted factorial (compare [49, p.38]) be defined by

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots, \end{cases} \tag{4}$$

The Watson notation [47] will also be used

$$(a;q)_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\ldots \end{cases} \tag{5}$$

Furthermore,

$$(a;q)_\infty \equiv \prod_{m=0}^{\infty}(1-aq^m),\ 0<|q|<1. \tag{6}$$

$$(a;q)_\alpha \equiv \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ a\neq q^{-m-\alpha}, m=0,1,\ldots. \tag{7}$$

Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k\langle 1;q\rangle_{n-k}}, \tag{8}$$

for $k=0,1,\ldots,n$, and by

$$\binom{\alpha}{\beta}_q \equiv \frac{\langle \beta+1,\alpha-\beta+1;q\rangle_\infty}{\langle 1,\alpha+1;q\rangle_\infty}, \tag{9}$$

for complex α and β when $0<|q|<1$.

The q-multinomial coefficient, a q-analogue of [120, p. 10], is defined by

$$\binom{n}{k_1,\ldots,k_l}_q \equiv \frac{\langle 1;q\rangle_n}{\prod_{i=1}^{l}\langle 1;q\rangle_{k_i}}, \tag{10}$$

for $\{k_i\}_{i=1}^l = 0,1,\ldots,n$ and $\sum_{i=1}^l k_i = n$.

If the number of k_i is unspecified, we denote the q-multinomial coefficient by

$$\binom{n}{\vec{k}}_q,\ \sum_{i=1}^{\infty} k_i = n. \tag{11}$$

We give some examples of q-multinomial coefficients.

Example 1.

$$\binom{3}{1,1,1}_q = 1+2q+2q^2+q^3. \tag{12}$$

$$\binom{4}{1,1,1,1}_q = 1+3q+5q^2+6q^3+5q^4+3q^5+q^6. \tag{13}$$

$$\binom{4}{2,1,1}_q = 1+2q+3q^2+3q^3+2q^4+q^5. \tag{14}$$

$$\binom{4}{2,2}_q = 1+q+2q^2+q^3+q^4. \tag{15}$$

Definition 3. If $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $E_q(z)$ was defined by Jackson [74] 1904, and by Exton [45].

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{16}$$

Euler found the following q-analogue of the exponential function

$$\mathrm{e}_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\langle 1; q\rangle_n} = \frac{1}{(z;q)_\infty}, \ |z| < 1, \ 0 < |q| < 1. \tag{17}$$

Let the q-Pochhammer symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} \equiv \prod_{m=0}^{n-1} \{a+m\}_q. \tag{18}$$

The following notation will be convenient.

$$\mathrm{QE}(x) \equiv q^x. \tag{19}$$

$$q^{\binom{\vec{k}}{2}} \equiv \prod_{j=1}^{n} q^{\binom{k_j}{2}}, \tag{20}$$

Definition 4. The Nalli–Ward–AlSalam q-addition (NWA), compare [5, p. 240], [96, p. 345], [141, p. 256], [29, p. 18] is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \ n = 0, 1, 2, \ldots. \tag{21}$$

Furthermore,we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k}, \ n = 0, 1, 2, \ldots. \tag{22}$$

There is a q-addition dual to the NWA, which will be presented here for reasons to be given shortly. The following polynomial in 3 variables x, y, q originates from Gauss.

Definition 5. The Jackson–Hahn–Cigler q-addition (JHC), compare [25, p. 91], [67, p. 362], [77, p. 78] is the function

$$\begin{aligned}(x \boxplus_q y)^n \equiv [x+y]_q^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} y^k x^{n-k} = \\ x^n(-\frac{y}{x};q)_n \equiv P_{n,q}(x,y), \ n = 0, 1, 2, \ldots.\end{aligned} \tag{23}$$

$$(x \boxminus_q y)^n \equiv P_{n,q}(x,-y), \ n = 0, 1, 2, \ldots. \tag{24}$$

Remark 1. The notation $[x+y]_q^n$ is due to Hahn [67, p. 362], and the notation $P_{n,q}(x,y)$ is due to Cigler [25, p. 91]. We will only use $(x \boxplus_q y)^n$ as it resembles the notation for NWA.

For symbolic purposes, we will define a general q-addition.

Definition 6. Let $f(k,n)$ be a given function. Then the general q-addition is defined by

$$(a \oplus_{g,q} b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{f(k,n)} a^k b^{n-k},\ n = 0, 1, 2, \ldots, \tag{25}$$

and similar for $(a \ominus_{g,q} b)$.

In chapter 3 we discuss how the NWA q-addition enables q-analogues of many results from Nörlunds investigations of difference analysis [101],[102]. We will use Milne-Thomson [94] as a basis for the notation of the various polynomials. The generating functions will play a major role here.

The main topic in chapters 3 and 4 is the set of almost parallel formulas, anticipated by Ward [141], for the q-Bernoulli, q-Euler, q-Lucas and q-G numbers and polynomials. These numbers will belong to $\mathbb{C}(q)$. In number theory there is also a Lucas number, which is not to be confused with this one. The reason for the name G-number is that E.T.Bell, and after him his graduate student Ward, for some reason called these numbers Genocchi numbers.

The reason for introducing the second or JHC polynomials is that they are needed in the q-analogues of complementary argument formulas. The notation second or JHC polynomials will be used throughout.

These equations are more systematically presented here than in [102], which makes this paper an amplification and a complement to [102] even for the case $q = 1$. Despite the "telegraphic style" of the proofs, which assume that the reader knows the basic technical tools and various q-identities, it is likely that readers with a taste for q-analogues will find much to enjoy. For example, the q-analogues of the Euler-Maclaurin summation formula might even be of some general interest.

We have added q-analogues of some formulas from Szegö's review [130] of [101]. The operational q-additions makes the formulas remarkably pretty.

It is well-known that there are at least two types of q-Bernoulli numbers, now let's consider the first one, i.e. NWA. Its complement is JHC. The different Carlitz' 1948 q-Bernoulli numbers are not considered here.

In the spirit of Milne-Thomson [94] and Rainville [108], we replace the $=$ by $\doteq$ to indicate the symbolic nature. Lucas [89] used a different symbol.

Example 2. [141, p. 265], [5, p. 245, 4.3],

$$B_{\mathrm{NWA},0,q} = 1,\ (B_{\mathrm{NWA},q} \oplus_q \ 1)^k - B_{\mathrm{NWA},k,q} \doteq \delta_{1,k}. \tag{26}$$

where $B^n_{\mathrm{NWA},q}$ is replaced by $B_{\mathrm{NWA},n,q}$ on expansion.

Another improvement in the present paper is that the operational umbral formulas for q-Bernoulli- and q-Euler polynomials are extended from polynomials to formal power series. The formulas in chapters 2,3 are adaptable to formal power series with corresponding q-Taylor formulae whereas the formulas in chapters 4,5 are adaptable to functions of q^x or equivalently $\binom{x}{k}_q$, with corresponding q-Taylor formulae.

In the year 1706 Johann Bernoulli (1667-1748) invented the difference symbol $\triangle$. Fifty years latter, 1755, Leonhard Euler used its inverse, the $\sum$ operator [44, chapter 1]. Euler was Johann Bernoulli's student together with Bernoulli's two sons, Nicolaus II and Daniel. Even though Johann Bernoulli used the symbol $\triangle$ already in 1706, he did not imply final differences thereby but differential quotients. Euler stands out as the one who devised the designation that has remained in use. Euler's proofs were however from a modern point of view not entirely satisfactory [70, p. 87].

When looking for original old references, and with no knowledge of Latin, it can be remarked that the calculation which lead Euler to define his $\sum$ operator can also be found in the books by Schweins [121, p. 9] and Goldstine [53, p. 98].

The two symbols, sometimes called the difference and sum calculus, correspond respectively to differentation and integration in the continuous calculus. We will find 2 different q-analogues of the inverse operators $\triangle$ and $\sum$ in chapters 2 and 4.

In the history of mathematics, Stirling numbers appeared in many different disguises. Cauchy used them in 1833 [22, p. 35] to compute sums of powers. Grünert [63] published an explicit formula and a recursion formula. Stirling numbers were also used in Saalschütz's book [114, p. 93].

Before Nielsen coined this name, the most frequent appearence of the so-called second Stirling number was as submultiple of the Euler formula for n-th differences of powers [58]. This formula is

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}i^n. \tag{27}$$

Tables of these numbers were published by De Morgan and Cayley [23]. The following review of Cayley's paper was written by Glaisher in Jahrbuch für Mathematik.

Grünert has made a table for $\frac{\Delta^m 0^n}{m!}$ until $m = n = 12$ in Crelle J. XXV. 279; Mr Cayley has checked this table and extended it to $m = n = 20$. This computational method is also analysed.

Glaisher himself published an even bigger table 1900.

A formula related to (27), which forms the basis for q-analysis is

$$\sum_{n=0}^{m}(-1)^n \binom{m}{n}_q q^{\binom{n}{2}} u^n = (u;q)_m. \tag{28}$$

According to Ward [141, p. 255] and Kuperschmidt [86, p. 244], this identity was first obtained by Euler. Gauss 1876 [48] also found this formula.

As will be seen, the corresponding expression for the second q-Stirling number has a slightly different character than (28).

An interesting discussion of several other notations for Stirling numbers is found in Knuth's essay [84]. This coincided with Kramp's introduction of $n!$ and Ettingshausens introduction of $\binom{m}{n}$ for binomial coefficients.

The 2 different q-analogues of $\triangle$ in chapters 3,4 combine with the same q-Stirling numbers. We will find many q-analogues of Stirling number identities from Jordan [81] and the elementary textbooks by Cigler [28] and Schwatt [119].

q-Stirling numbers are of the greatest benefit in q-calculus. This however has not been fully acknowledged until now. In a book by Don Knuth [85], it is shown that the q-Stirling number of the second kind gives the running time of the algorithm for a computer program. A related result for Markov processes was obtained by Crippa, D.; Simon, K.; Trunz, P. [32].

As Sharma and Chak [118, p. 326] remarked, the operator D_q, defined by

$$(D_q\varphi)(x) \equiv \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \frac{d\varphi}{dx}(x) & \text{if } q = 1; \\ \frac{d\varphi}{dx}(0) & \text{if } x = 0 \end{cases} \tag{29}$$

plays the same role for polynomials in x as the difference operator in chapter 4

$$\triangle_q f(x) \equiv f(x+1) - f(x),\ \triangle_q^{n+1} f(x) \equiv \triangle_q^n f(x+1) - q^n \triangle_q^n f(x) \tag{30}$$

does for polynomials in q^x.

If we want to indicate the variable which the q–difference operator is applied to, we write $(D_{q,x}\varphi)(x,y)$ for the operator. The same notation will also be used for a general operator.

The Hahn operator in chapter 3

$$\triangle_{H,q} f(x) \equiv \frac{f(qx+1)-f(x)}{1+(q-1)x} \tag{31}$$

will prove useful in connection with q-analogues of Eulers equations for Stirling numbers.

In chapter 5 we treat another q-difference operator

$$\triangle_{\mathrm{J},q} f(q^x) \equiv (f(q^{x+1})-f(q^x))q^{-x}. \tag{32}$$

In the last chapter we give some applications to q-binomial coefficient identities.

In each of chapters 2,3,4,5 for the respective $\triangle_q$ operator; and in chapter two ∇_q, we will find q-analogues of Leibniz-type formulas from Jordan [81].

In the footsteps of Faulhaber and Fermat, we will find a q-analogue of (1) in chapter 2, which extends naturally to generalizations in the spirit of Lucas. In chapters 3 and 4 we will use the Carlitz function

$$S_{\mathrm{C},m,q}(n) \equiv \sum_{i=0}^{n-1} \{i\}_q^m q^i \tag{33}$$

to find q-analogues of Munch [95], which are more general than the recent paper by Schlosser [117].

All the next 3 equations were found by Euler. They have the following form, where E is the forward shift operator and $\triangle = E - I$.

Theorem 1.1. [43, p. 200], [28, p. 26], [121, p. 9].

$$\triangle^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{n-k} f(x). \tag{34}$$

This formula can be inverted.

Theorem 1.2. [122, p. 15, 3.1]

$$E^n f(x) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f(x). \tag{35}$$

The Leibniz rule is as follows.

Theorem 1.3. [81, p. 97, 10], [28, p. 27, 2.13], [94, p. 35, 2], [103, p. 19].

$$\triangle^n(fg) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f \ (\triangle^{n-i} E^i)) g. \tag{36}$$

Curiously, in chapter 2 we keep the binomial coefficients in the corresponding q-formulas, whereas in the remaining 3 chapters, the q-binomial coefficients are used for the corresponding formulas. Assume that we choose a difference operator $\triangle_q$ from chapter 3,4 or 5 and operate on the q-shifted factorial. The result is

$$\triangle_{x,q}^n \langle x+y;q\rangle_k = \{k-n+1\}_{n,q}\langle x+y+n;q\rangle_{k-n}\mathrm{QE}(f(x,y,k,n)), \quad (37)$$

with proper normalization. This implies

$$\triangle_{x,q}^n \binom{x}{k}_q = \binom{x}{k-n}_q \mathrm{QE}(g(x,k,n)). \quad (38)$$

As $f(x,y,k,n)$ and $g(x,k,n)$ are independent of x for $\triangle_{\mathrm{J},q}$ in chapter 5, this operator is preferable in calculations with q-binomial coefficents and q-shifted factorials.

2. The Ward-Alsalam-Rota-Cigler q-umbral calculus

We first give a short review of the history of umbral calculus and finite differences. Interpolation theory, often used by nineteenth century astronomers (Gauss, Bessel, W. Herschel [1738–1822], J. Herschel), is essentially equivalent to theory of finite differences. Gudermann (1786–1852), the teacher of Weierstrass, was one of the first to use this technique. Calculations on elliptic functions by finite differences were made by Jacobi, Weierstrass and Milne-Thomson (1891–1974). The mathematician and astronomer Johann August Grünert (1797–1872), editor of the journal *Archiv der Mathematik und Physik*, which started in 1841, used this technique to publish some of the first tables of Stirling numbers. The umbral calculus was initiated by Euler [44], who used operator equations like (149), and Lagrange (1736–1813). Later Arbogast (1759–1803) suggested to substitute a capital D for the little d of Leibniz to simplify the computations. Textbooks on the subject were written by Ettingshausen (1796–1878), J. Herschel (1792–1871), Pearson 1850 and De Morgan (1806–1871). Robert Murphy (1806–1843) was a forerunner to Boole and Heaviside, who among other things found beautiful operator formulas for derivatives in the spirit of Carlitz. J.J. Sylvester (1814–1897) edited *Quarterly Journal of Mathematics* from 1855 to 1878, where attempts at umbral calculus were made by Horner 1861, Blissard (1803–1875) 1861-68, and Glaisher (1848–1928). It was Sylvester who coined the name umbral calculus. By 1860 two textbooks on finite differences were in print in England, one of them by Boole (1815–1864), which covered almost all the theorems that we know now. Heaviside (1850–1925) was able to greatly simplify Maxwell's 20 equations in 20 variables to four equations in two variables. This and other articles about electrical problems, which appeared in 1892-98, were severely criticized for their lack of rigour by the contemporary mathematicians. Genocchi (1817–1889) and Pincherle (1853–1936) contributed to the early Italian development of the subject. Clebsch (1833–1872) and Gordan (1837–1912) continued the theory of invariants that had started with Sylvester and Cayley.

In the same way as the Γ function plays a basic role in complex analysis, the Γ_q function is fundamental for q-calculus. The Γ_q function is defined in the unit disk $0 < |q| < 1$ by

Definition 7.

$$\Gamma_q(x) \equiv \frac{\langle 1; q\rangle_\infty}{\langle x; q\rangle_\infty}(1-q)^{1-x}. \tag{39}$$

Heine, P. Appell 1879, and Daum [34] used another function without the factor $(1-q)^{1-x}$, which they called the Heine Ω-function. Ashton [11] in his thesis supervised by Lindemann, showed its connection to the Jacobi-Neville elliptic functions. The main difference between the two functions is that Ω has zeros, in contrast to the Γ_q function which has no zeros, and therefore $\frac{1}{\Gamma_q}$ is entire. Sonine [125] wrote a book about the Heine Ω-function in Russian.

In 1880 P. Appell (1855–1930) [9] characterized certain polynomial sequences $F_\nu(x)$ including Bernoulli and Euler by the property

$$DF_\nu(x) = \nu F_{\nu-1}(x). \tag{40}$$

This was equivalent to the Euler generating function. Another French contribution was made by E. Lucas (1842–1891), who invented a modern notation for umbral calculus, which we will follow closely. F.H. Jackson (1870–1960) followed this path in the early twentieth century, and fully understood the symbolic nature of the subject in his first investigations of q-functions. Like Blissard, Jackson worked as a priest his whole life; both of them had studied in Cambridge. To commemorate Jackson, we will use his notation for $E_q(x)$.

Pia Nalli (1886–1964) was the first to use the Ward q-addition in her only paper on q-calculus. Letterio Toscano published interesting papers involving Bernoulli, Euler and Stirling numbers in connection with the operator xD. Geronimus (1898-) wrote about certain Appell polynomials. E–T. Bell (1883-1960) tried to write about umbral calculus, but he is best rememberd for his books about the history of mathematics. One of the few q-analysts in America in the beginning of the twentieth century was Morgan Ward (1901–1963), who became doctor at Caltech 1928 supervised by E. T. Bell. He worked in Princeton 1934-35 together with Siegel on the analytic theory of quadratic forms. Back in Caltech 1936, Ward supervised Robert Dilworth's doctoral studies.

Thorvald N. Thiele (1838–1910) was a prolific Danish actuary, astronomer, and mathematician with an impressive record. Thiele's book Interpolationsrechnung, which contains a table of Stirling numbers, was published 1909.

Niels Erik Nörlund (1885–1981) was a Danish-Swedish mathematician and geodeticist. After a start as an astronomer in Copenhagen 1908-1912, one year after Thiele's retirement as director of the observatory 1907, Nörlund went to Lund and later became editor of *Acta Math*. The remarkable work [102] presented the first rigorous treatment of finite differences, written from the point of view of the mathematician. According to Grigoriew [62, p.147], the generalized Bernoulli numbers that Nörlund uses in [102] had previously also been used by Blissard [14] and Imchenetsky [73].

Steffensen [129], Jordan [81], and Milne-Thomson [94] wrote books about finite differences intended both for mathematicians and statisticians. Johann Cigler (1937-) [28] wrote an excellent book on finite differences with a view to umbral calculus. The Heine q-umbral calculus reached its peak in the thesis by Smith [124] 1911, supervised by Pringsheim. The Austrian school of q-analysis had started already in the sixties when Wolfgang Hahn (1911-1998) moved to Graz in 1964 after visits to India 1959-1961 and America 1962. Other famous people are L. Carlitz (1907–1999), J. Riordan, and Rota (1932-1999). In [27] a special case of the following q-umbral calculus was used, the case $q = 1$ was treated in [28].

Definition 8. A q-analogue of [113, p. 696]. A q-umbral calculus contains a set A, called the alphabet, with elements called letters or umbrae.

Assume that α, β, are distinct umbrae, then a new umbra is obtained by $\alpha * \beta$, where $*$ is $\oplus_q$, $\boxplus_q$, $\ominus_q$, $\boxminus_q$, or any general q-addition.

There is a certain linear functional $eval$, $\mathbb{C}[[x]] \times A \to \mathbb{C}$, called the evaluation, such that $eval(1) = 1$.

If $\alpha,\ \beta,\ \ldots,\ \gamma$ are distinct umbrae, and $i, j,\ \ldots,\ k$ positive integers,

$$eval(\alpha^i \beta^j \ldots \gamma^k) = eval(\alpha^i) eval(\beta^j) \ldots eval(\gamma^k). \tag{41}$$

Two umbrae α and β are said to be equivalent, denoted $\alpha \sim \beta$ if $eval(\alpha) = eval(\beta)$. The set of equivalent umbrae form an equivalence class.

There is a distinguished element ϵ of the alphabet called the zero, such that

$$eval(\epsilon^n) = \delta_{n,0} \quad \text{and} \quad x \boxminus_q x \sim \epsilon. \tag{42}$$

Elements α and $\beta \in A$ are said to be inverse to each other if $\alpha \boxplus_q \beta \sim \epsilon$.

There is a Ward number $\overline{n}_q$

$$\overline{n}_q \sim 1 \oplus_q 1 \oplus_q \ldots \oplus_q 1, \tag{43}$$

where the number of 1 in the RHS is n.

There is a Jackson number $\tilde{n}_q$

$$\tilde{n}_q \sim 1 \boxplus_q 1 \boxplus_q \ldots \boxplus_q 1, \tag{44}$$

where the number of 1 in the RHS is n.

There is a multiplication with Ward- and Jackson numbers in the following sense: Assume that $a \in \mathbb{C}$ or $a \in \mathbb{C}[D_q]$. Then we define

$$a\overline{n}_q \sim a \oplus_q a \oplus_q \ldots \oplus_q a, \tag{45}$$

where the number of a in the RHS is n. In the same way,

$$a\tilde{n}_q \sim a \boxplus_q a \boxplus_q \ldots \boxplus_q a, \tag{46}$$

where the number of a in the RHS is n.

If $\alpha_1, \ldots, \alpha_n \in A,\ \alpha \sim \alpha_i,\ i = 1, \ldots, n$ then $\alpha_1 \oplus_q \ldots \oplus_q \alpha_n \sim \overline{n}_q \alpha$. The last condition is a q-analogue of [101, p. 125, (13)], [102, p. 132, (49)].

$$F(x \oplus_q B_{\text{NWA},q}(y)) \ddot{=} \sum_{n=0}^{\infty} \frac{B_{\text{NWA},n,q}(y)}{\{n\}_q!} D_q^n F(x). \tag{47}$$

Here B can be changed to any q-polynomial sequence.

Three examples of *eval* are the NWA, JHC and the general q-addition.

Theorem 2.1. [96, p. 345]. *The q-addition (21) has the following properties, $a, b, c \in A$:*

$$\begin{aligned}
&(a \oplus_q b) \oplus_q c \sim a \oplus_q (b \oplus_q c)\\
&a \oplus_q b \sim b \oplus_q a\\
&a \oplus_q 0 \sim 0 \oplus_q a \sim a\\
&ca \oplus_q cb \sim c(a \oplus_q b).
\end{aligned} \tag{48}$$

The first three conditions means that the umbrae of Nalli-Ward-Alsalam q-addition form a commutative monoid.

Definition 9. For $\alpha \in \mathbb{C}$, NWA is extended to

$$(a \oplus_q b)^\alpha \equiv a^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k}_q (\frac{b}{a})^k, \ |\frac{b}{a}| < 1. \tag{49}$$

Remark 2. The associative law doesn't hold here.

Definition 10. For $\alpha \in \mathbb{C}$, JHC is extended to

$$(a \boxplus_q b)^\alpha \equiv a^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k}_q (\frac{b}{a})^k q^{\binom{k}{2}}, \ |\frac{b}{a}| < 1. \tag{50}$$

We will give three examples of other scientists who have used these q-additions in other contexts.

In 1954 Sharma A. & Chak A. M. [118] constructed q-Appell sequences for the JHC. The JHC has also been used by Goulden & D.M. Jackson [60], who used the notation

$$Q_n(-y, x) \equiv P_{n,q}(x, y).$$

In 1994 [30] Chung K. S. & Chung W. S. & Nam S. T. & Kang H. J. rediscovered the NWA together with a new form of the q-derivative.

Definition 11. A generalization of [30].

Let $*$ be $\oplus_q$ or $\boxplus_q$. Then

$$D_* f(x) = \lim_{\delta x \to 0} \frac{f(x * \delta x) - f(x)}{\delta x}. \tag{51}$$

Theorem 2.2. *A generalization of* [40]. *Let* $*$ *be* $\oplus_q$ or $\boxplus_q$. *Then the two operators* D_q *and* D_* *are identical when operating on functions which can be expressed as* $x^\alpha \sum_{k=0}^{\infty} a_k x^k, \ \alpha \in \mathbb{C}$.

Unlike the NWA, the JHC is neither commutative nor associative, but on the other hand, it can be written as a finite product.

Jackson and Exton have presented several addition theorems for q-exponential and q-trigonometric functions. These are presented in a more lucid style using the JHC q-addition in [39].

Ward explained how the NWA can be used as the function argument in a formal power series. The theory of formal power series is outlined in Niven [100], see also Hofbauer [71].

The formal power series form a vector space with respect to termwise addition and multiplication by complex scalars. In the rest of this chapter, as in [141, p. 258], unless otherwise stated, we assume that functions $f(x), g(x), F(x), G(x) \in \mathbb{C}[[x]]$.

Definition 12. If

$$F(x)=\sum_{k=0}^{\infty}a_kx^k, \tag{52}$$

[141, p. 258]

$$F(x\oplus_q y)\equiv\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\binom{n}{k}_q x^ky^{n-k}. \tag{53}$$

[77, p. 78]

$$F(x\boxplus_q y)\equiv F[x+y]_q\equiv\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}\, y^kx^{n-k}. \tag{54}$$

In 1936 Ward [141, p. 256] proved the following equations for q-subtraction
(The original paper seems to contain a misprint of (55).):

$$(x\ominus_q y)^{2n+1}=\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}_q x^ky^k(x^{2n+1-2k}-y^{2n+1-2k}). \tag{55}$$

$$(x\ominus_q y)^{2n}=(-1)^n\binom{2n}{n}_q x^ny^n+\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}_q x^ky^k(x^{2n-2k}+y^{2n-2k}). \tag{56}$$

We could also use norms for q-additions. To this aim we put

Definition 13. The norm for NWA is defined by

$$|a\oplus_q b|_n\equiv|(a\oplus_q b)^n|^{\frac{1}{n}}. \tag{57}$$

Assume that we would like to use the q-binomial theorem for

$$\frac{1}{((a\oplus_q b);q)_\alpha}, \tag{58}$$

where $(a\oplus_q b)^n$ is defined by (21) and $a,b\in A$. To use the q-binomial theorem we should require $|a\oplus_q b|_n<1$, for $n>0$.

Before we embark on q-Taylor theorems, the following remark of Pearson [103] might be of interest. The differential calculus is a particular case of the direct method of finite differences, and the integral calculus is a particular case of the inverse method of finite differences. In fact Taylor published his formula in terms of finite differences.

There are at least three q-analogues of the Taylor formula for formal power series known from the literature, which we will list here. Compare [100, p. 877] for reference on formal power series.

Theorem 2.3. *The Nalli–Ward q-Taylor formula.* [96, p. 345], [141, p. 259].

$$F(x\oplus_q y)=\sum_{n=0}^{\infty}\frac{y^n}{\{n\}_q!}D_q^nF(x). \tag{59}$$

Theorem 2.4. *The first Jackson q-Taylor formula* [75, p. 63].

$$F(x)=\sum_{n=0}^{\infty}\frac{(x\boxminus_q y)^n}{\{n\}_q!}D_q^nF(y). \tag{60}$$

Theorem 2.5. *The second Jackson q-Taylor formula* [77, (51, p.77)]

$$F(x\boxplus_q y)=\sum_{n=0}^{\infty}\frac{y^n}{\{n\}_q!}q^{\binom{n}{2}}D_q^nF(x). \tag{61}$$

Remark 3. Wallisser [140] has found a criterion for an entire function to be expanded in the q-Taylor series (60) for the special case $y=1$ and $q<1$. Put $M_{\mathrm{E}_{\frac{1}{q}}}(r)=\max_{|x|=r}|\mathrm{E}_{\frac{1}{q}}(x)|$.

If the maximum of the absolute value of an entire function F on $|x|=r$ satisfies the inequality

$$M_F(r)\leq CM_{\mathrm{E}_{\frac{1}{q}}}(r\tau),\ q\in\mathbb{R},\ q<1,\ \tau<(\frac{1}{q}-1)^{-1}, \tag{62}$$

then $F(x)$ can be expanded in the q-Taylor series (60) for the special case $y=1$.

Remark 4. Schendel [116] first proved (60), possibly influenced by Gauss.

The following general inversion formula will prove useful in the sequel.

Theorem 2.6. *Gauss inversion* [4, p. 96]*, a corrected version of* [52, p. 244]*. A q-analogue of* [109, p. 4]*. The following two equations for arbitrary secuences $a_n,\ b_n$ are equivalent.*

$$a_n=q^{-f(n)}\sum_{l=0}^{n}(-1)^lq^{\binom{l}{2}}\binom{n}{l}_q b_{n-l}, \tag{63}$$

$$b_n=\sum_{i=0}^{n}q^{f(i)}\binom{n}{i}_q a_i, \tag{64}$$

Proof. It will suffice to prove that

$$a_n=q^{-f(n)}\sum_{l=0}^{n}(-1)^l\binom{n}{l}_q q^{\binom{l}{2}}\sum_{i=0}^{n-l}q^{f(i)}\binom{n-l}{i}_q a_i. \tag{65}$$

The first sum is zero except for $i=n$ and $l=0$. □

Definition 14. The Ward-Alsalam q-shift operator [5, p. 242, 3.1], a q-analogue of [16, p. 16], [28, p. 18], [102, p. 3], is given by

$$E(\oplus_q)^{\omega}(x^n)\equiv(x\oplus_q\omega)^n \tag{66}$$

We denote the corresponding operator for the JHC by $E(\boxplus_q)$. i.e.

$$E(\boxplus_q)^{\omega}(x^n)\equiv(x\boxplus_q\omega)^n \tag{67}$$

When $\omega=1$, we denote these operators $E(\oplus_q)$ and $E(\boxplus_q)$.

Definition 15. The invertible linear difference operator for the NWA, a q-analogue of [102, p. 3], is defined by

$$\underset{\omega}{\triangle_{\text{NWA},q}} \equiv \frac{E(\oplus_q)^{\omega} - I}{\omega},\ \omega \in \mathbb{C}, \tag{68}$$

where I is the identity operator. When $\omega = 1$, we denote this operator $\triangle_{\text{NWA},q}$ [5, p. 243, 3.5], compare [141, p. 264, 15.1].

Remark 5. In contrast to [94], $\lim_{\omega \to 0} \underset{\omega}{\triangle_{\text{NWA},q}}$ does not correspond to a q-difference operator.

Definition 16. The linear difference operator for the JHC, a q-analogue of [102, p. 3], is defined by

$$\underset{\omega}{\triangle_{\text{JHC},q}} \equiv \frac{E(\boxplus_q)^{\omega} - I}{\omega},\ \omega \in \mathbb{C}, \tag{69}$$

where I is the identity operator. When $\omega = 1$, we denote this operator $\triangle_{\text{JHC},q}$.

Definition 17. If ω is a Ward number $\overline{n}_q$, the difference operator for the NWA is defined by

$$\underset{\overline{n}_q}{\triangle_{\text{NWA},q}} \equiv \frac{E(\oplus_q)^{\overline{n}_q} - I}{n}. \tag{70}$$

Definition 18. If ω is a Jackson number $\tilde{n}_q$, the difference operator for the JHC is defined by

$$\underset{\tilde{n}_q}{\triangle_{\text{JHC},q}} \equiv \frac{E(\boxplus_q)^{\tilde{n}_q} - I}{n}. \tag{71}$$

Remark 6. The formulas (82) and (101) show that the minus between $E(\oplus_q)$ and I is not a q-subtraction.

We are now going to present an operational equation, which was first found by Lagrange 1772 for $q = 1$. It played a major role in the theory of finite differences, for example in Lacroix's treatise on differences from 1800, [10],[16, p. 18], [69, p. 26], [70, p. 66]; and later in the first umbral calculus by Blissard [14]. Cauchy also used it in 1844 [22]. The following dual q-analogues of [28, p. 28], see [5, p. 242, 3.3, p. 243, 3.9], [141, p. 264] hold.

$$E(\oplus_q)^{\omega} = \mathrm{E}_q(\omega D_q). \tag{72}$$

$$E(\boxplus_q)^{\omega} = \mathrm{E}_{\frac{1}{q}}(\omega D_q). \tag{73}$$

The difference operator $\triangle$ was used by Boole [16, p. 16], who showed that $\triangle$ is distributive, commutative with respect to any constant coefficients in the terms of the object to which it is applied, and obeys the index law for exponents. The same laws obtain for $\triangle_{\mathrm{NWA},q}$ and for the operator $\triangle_{\mathrm{J},q}$, but not for $\triangle_{\mathrm{CG},q}$.

Definition 19. A q-analogue of the mean value operator of Jordan [81, p. 6] ($\omega = 1$), Nörlund [102, p. 3], and [94, p. 30].

$$\underset{\omega}{\nabla_{\mathrm{NWA},q}} \equiv \frac{E(\oplus_q)^{\omega} + I}{2}. \tag{74}$$

When $\omega = 1$, we denote this operator $\nabla_{\mathrm{NWA},q}$.

Definition 20. A JHC q-analogue of the mean value operator of Jordan [81, p. 6] ($\omega = 1$), Nörlund [102, p. 3], and [94, p. 30].

$$\underset{\omega}{\nabla_{\mathrm{JHC},q}} \equiv \frac{E(\boxplus_q)^{\omega} + I}{2}. \tag{75}$$

When $\omega = 1$, we denote this operator $\nabla_{\mathrm{JHC},q}$.

In the following definition, the q-additions are written first in additive, then in multiplicative form. In the first case, we assume that the function argument operate from left to right when using the two q-additions. In the second case, we assume that the function argument operate from right to left in accordance with (72) and (73). So don't forget that the following two equations are not associative.

Definition 21. If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{76}$$

$$f(x \oplus_q y \boxminus_q z) \equiv \\ \sum_{k=0}^{\infty} a_k \sum_{l=0}^{k} \binom{k}{l}_q y^{k-l} \sum_{m=0}^{l} (-1)^m \binom{l}{m}_q q^{\binom{m}{2}} z^m x^{l-m}. \tag{77}$$

$$f(E(\boxplus_q)^{-z} E(\oplus_q)^y x)) \equiv \\ \sum_{k=0}^{\infty} a_k \sum_{l=0}^{k} \binom{k}{l}_q y^{k-l} \sum_{m=0}^{l} (-1)^m \binom{l}{m}_q q^{\binom{m}{2}} z^m x^{l-m}. \tag{78}$$

We will now give a number of theorems for arbitrary letters which illustrate certain symmetry properties of this umbral calculus.

Theorem 2.7. *The NWA and the JHC are dual operators.*

$$f(x \oplus_q a \boxminus_q a) \equiv f(E(\boxplus_q)^{-a} E(\oplus_q)^a x) = f(x). \tag{79}$$

Proof. Use (72) and (73). □

By Goulden & D.M. Jackson [60] we obtain two further formulas of this type.

Theorem 2.8. [60, p. 228]

$$(\alpha \boxplus_q \beta) \oplus_q (\gamma \boxplus_q \delta) \sim (\alpha \boxplus_q \delta) \oplus_q (\gamma \boxplus_q \beta),\quad \alpha,\beta,\gamma,\delta \in A. \tag{80}$$

Theorem 2.9. [60, p. 228]

$$(\alpha \boxminus_q \gamma) \sim (\alpha \boxminus_q \beta) \oplus_q (\beta \boxminus_q \gamma),\quad \alpha,\beta,\gamma \in A. \tag{81}$$

The 2 Leibniz theorems go as follows. Notice the binomial coefficient on the right.

Theorem 2.10. *A q-analogue of* [28, p. 27, 2.13], [81, p. 97, 10], [94, p. 35, 2] *. Let $f(x)$ and $g(x)$ be formal power series. Then*

$$\triangle_{\mathrm{NWA},q}^{n}(fg) = \sum_{i=0}^{n} \binom{n}{i} \triangle_{\mathrm{NWA},q}^{i} f(\triangle_{\mathrm{NWA},q}^{n-i} E(\oplus_q)^i) g. \tag{82}$$

Proof. Same as [81, p. 96 f]. □

Theorem 2.11. *A q-analogue of* [28, p. 27, 2.13], [81, p. 97, 10], [94, p. 35, 2] *. Let $f(x)$ and $g(x)$ be formal power series. Then*

$$\triangle_{\mathrm{JHC},q}^{n}(fg) = \sum_{i=0}^{n} \binom{n}{i} \triangle_{\mathrm{JHC},q}^{i} f(\triangle_{\mathrm{JHC},q}^{n-i} E(\boxplus_q)^i) g. \tag{83}$$

The difference of a quotient of functions can be computed as

Theorem 2.12. *A q-analogue of* [103, p. 2], [16, p. 29].

$$\triangle_{\mathrm{NWA},q} \frac{f(x)}{g(x)} = \frac{g(x)\triangle_{\mathrm{NWA},q} f(x) - f(x)\triangle_{\mathrm{NWA},q} g(x)}{g(x) g(x \oplus_q 1)}. \tag{84}$$

Theorem 2.13. *A q-analogue of* [103, p. 2], [16, p. 29].

$$\triangle_{\mathrm{JHC},q} \frac{f(x)}{g(x)} = \frac{g(x)\triangle_{\mathrm{JHC},q} f(x) - f(x)\triangle_{\mathrm{JHC},q} g(x)}{g(x) g(x \boxplus_q 1)}. \tag{85}$$

The following theorem reminding of [141, p. 258] shows how Ward numbers usually appear in applications. Compare with [5, p. 244, 3.16], where the notation $P_k(n)$ was used.

Theorem 2.14.

$$(\overline{n}_q)^k = \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1,\ldots,m_n}_q, \tag{86}$$

where each partition of k is multiplied with its number of permutations. We have the following special cases:

$$(\overline{0}_q)^k = \delta_{k,0};\ (\overline{n}_q)^0 = 1;\ (\overline{n}_q)^1 = n. \tag{87}$$

The following theorem shows how Jackson numbers usually appear in applications.

Theorem 2.15.

$$(\tilde{n}_q)^k = \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1,\ldots,m_n}_q q^{\binom{\vec{m}}{2}}, \quad \vec{m}=(m_2,\ldots,m_n), \tag{88}$$

where for each partition of k, all permutations are counted. We have the following special cases:

$$(\tilde{0}_q)^k = \delta_{k,0};\ (\tilde{n}_q)^0 = 1;\ (\tilde{n}_q)^1 = n. \tag{89}$$

The following table lists some of the first $(\overline{n}_q)^k$. Compare [31, p. 309], where a long list of multinomial coefficients is given. The reader can check that the results agree with the definition (86) for the case $q = 1$.

	$k=2$	$k=3$	$k=4$
$n=1$	1	1	1
$n=2$	$3+q$	$4+2q+2q^2$	$5+3q+4q^2+3q^3+q^4$
$n=3$	$6+3q$	$10+8q+8q^2+q^3$	$3\,(5+5q+7q^2+6q^3+3q^4+q^5)$
$n=4$	$10+6q$	$4\,(5+5q+5q^2+q^3)$	$5\,(7+9q+13q^2+12q^3+7q^4+3q^5)+q^6$

The following table lists some of the first $(\tilde{n}_q)^k$.

	$k=2$	$k=3$	$k=4$
$n=1$	1	1	1
$n=2$	$2+2q$	$2(1+q+q^2+q^3)$	$2+2q+2q^2+3q^3+2q^4+2q^5+q^6$
$n=3$	$4+5q$	$4+8q+8q^2+7q^3$	$(2+2q+2q^2+3q^3)^2$
$n=4$	$7+9q$	$4\,(2+5q+5q^2+4q^3)$	$8+24q+41q^2+63q^3+56q^4+39q^5+25q^6$

According to Netto [97], the so-called multinomial expansion theorem was first mentioned in a letter 1695 from Leibniz to Johann Bernoulli, who proved it. In 1698 De Moivre first published a paper about multinomial coefficients in England [36, p. 114].

Two natural q-analogues are given by

Definition 22. If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, it's k'th NWA-power is given by

$$(\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q. \tag{90}$$

Definition 23. If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, it's k'th JHC-power is given by

$$(\boxplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \boxplus_q a_1 x \boxplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q q^{\binom{\vec{n}}{2}}, \tag{91}$$

where $\vec{n} = (m_2, \ldots, m_n)$.

Definition 24. If $f(x)$ is the formal power series $\sum_{k=0}^{\infty} a_k x^k$, the (Ward) q-sum is defined by

$$\sum_{k=n}^{m} f(\overline{k_q}) \equiv \sum_{k=n}^{m} \sum_{l=0}^{\infty} a_l (\overline{k}_q)^l, \; n, m \in \mathbb{N}, \; n \le m, \tag{92}$$

where for each k the function value for the corresponding Ward number is computed. If $n > m$, the sum $= 0$. Similarly, we define

$$(-1)^{\overline{n_q}} \equiv (-1)^n. \tag{93}$$

Definition 25. If $f(x)$ is the formal power series $\sum_{k=0}^{\infty} a_k x^k$, the (Jackson) q-sum is defined by

$$\sum_{k=n}^{m} f(\tilde{k_q}) \equiv \sum_{k=n}^{m} \sum_{l=0}^{\infty} a_l (\tilde{k}_q)^l, \; n, m \in \mathbb{N}, \; n \le m, \tag{94}$$

where for each k the function value for the corresponding Jackson number is computed. If $n > m$, the sum $= 0$. Similarly, we define

$$(-1)^{\tilde{n_q}} \equiv (-1)^n. \tag{95}$$

Definition 26. Let the q-extended real numbers $\mathbb{R}_q$ be the set generated by $\mathbb{R}$ together with the operation $\oplus_q$, where $0 < q < 1$.

Definition 27. The real q-integral is defined by

$$\int_0^a f(t, q)\, d_q(t) \equiv a(1-q) \sum_{n=0}^{\infty} f(aq^n, q) q^n, \; 0 < q < 1, \; a \in \mathbb{R}_q. \tag{96}$$

Definition 28. Let $a = a(q) \in \mathbb{R}_q$ and $\lim_{q \to 1} a(q) > 0$. Then we can define a q-analogue of a closed interval as usual as [0,a].

Definition 29. Let $a \in \mathbb{R}_q$ and let I be the product interval $[0,a] \times (0,1)$. Then $\mathrm{L}_q^1(I)$ is the space of all functions $f(x,q) \in \mathbb{R}[[x]]$ on I such that

$$\int_0^a f(t,q)\, d_q(t) \tag{97}$$

converges.

Theorem 2.16. $\mathrm{L}_q^1(I)$ *is a vector space with respect to term-wise addition and multiplication by complex scalars.*

Theorem 2.17. *A q-analogue of the Newton-Gregory series* [28, p. 21, 2.7], [81, p. 26], [94, 2.5.1], [89, p. 243].

$$f(\overline{n}_q) = \sum_{k=0}^{n} \binom{n}{k} \triangle_{\mathrm{NWA},q}^{k} f(0). \tag{98}$$

This can be generalized to

Theorem 2.18. *A q-analogue of* [102, p. 4, (7)]

$$f(\omega\overline{n}_q) = \sum_{k=0}^{n} \binom{n}{k} \omega^k \left(\frac{\triangle_{\mathrm{NWA},q}}{\omega} \right)^k f(0). \tag{99}$$

Theorem 2.19. *A q-analogue of* [102, p. 4, (8)]

$$f(\omega\overline{n}_q) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^k \left(\frac{\nabla_{\mathrm{NWA},q}}{\omega} \right)^k f(0). \tag{100}$$

The formula (98) can be inverted as follows

Theorem 2.20. *A corrected form of* [141, p. 264, (iii)], *and a q-analogue of* [61, p. 188, (5.50)], [89, p. 136, (3)], [94, 2.5.2].

$$\triangle_{\mathrm{NWA},q}^{n} f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x \oplus_q \overline{k}_q). \tag{101}$$

We will use the following abbreviations, $k_l \in \mathbb{N}$.

$$\Omega \equiv (k_1\omega_1 \oplus_q k_2\omega_2 \oplus_q \ldots \oplus_q k_n\omega_n),\ k \equiv \sum_{l=1}^{n} k_l. \tag{102}$$

$$\Phi \equiv (\overline{k_1 m_{1q}}) \oplus_q (\overline{k_2 m_{2q}}) \oplus_q \ldots \oplus_q (\overline{k_n m_{nq}}). \tag{103}$$

The notation $\sum_{\vec{k}}$ denotes a multiple summation with each of the indices $k_1, \ldots, k_n$ running between $0, 1$.

The formula (101) can be generalized to

Definition 30. Two q-analogues of [102, p. 4]

$$\begin{matrix}\triangle^n_{\mathrm{NWA},q}\\ \omega_1,\ldots,\omega_n\end{matrix} f(x) \equiv (\omega_1\ldots\omega_n)^{-1}\sum_{\vec{k}}(-1)^{n-k}f(x\oplus_q \Omega). \tag{104}$$

$$\begin{matrix}\triangle^n_{\mathrm{NWA},q}\\ \overline{m_1}_q,\ldots,\overline{m_n}_q\end{matrix} f(x) \equiv (m_1\ldots m_n)^{-1}\sum_{\vec{k}}(-1)^{n-k}f(x\oplus_q \Phi). \tag{105}$$

There is a similar formula for $\nabla_{\mathrm{NWA},q}$.

$$\nabla^n_{\mathrm{NWA},q}f(x)\equiv 2^{-n}\sum_{k=0}^{n}\binom{n}{k}f(x\oplus_q \overline{k}_q). \tag{106}$$

In a similar way, the formula (106) can be generalized to

Definition 31. A q-analogue of [102, p. 4]

$$\begin{matrix}\nabla^n_{\mathrm{NWA},q}\\ \omega_1,\ldots,\omega_n\end{matrix} f(x)\equiv 2^{-n}\sum_{\vec{k}}f(x\oplus_q \Omega). \tag{107}$$

Theorem 2.21. *A q-analogue of* [81, (12), p. 114].

$$\nabla^{-1}_{\mathrm{NWA},q}=\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m}\triangle^m_{\mathrm{NWA},q}. \tag{108}$$

We will now find several q-analogues of formulas by Nörlund et. al. for difference operators. Some of them have been published before.

Theorem 2.22. *A q-analogue of the Newton-Gregory series* [28, p. 21, 2.7], [81, p. 26], [94, 2.5.1], [89, p. 243].

$$f(\tilde{n}_q)=\sum_{k=0}^{n}\binom{n}{k}\triangle^k_{\mathrm{JHC},q}f(0). \tag{109}$$

This can be generalized to

Theorem 2.23. *A q-analogue of* [102, p. 4, (7)]

$$f(\omega\tilde{n}_q)=\sum_{k=0}^{n}\binom{n}{k}\omega^k\left(\begin{matrix}\triangle_{\mathrm{JHC},q}\\ \omega\end{matrix}\right)^k f(0). \tag{110}$$

Theorem 2.24. *A q-analogue of* [102, p. 4, (8)]

$$f(\omega\tilde{n}_q)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^k\left(\begin{matrix}\nabla_{\mathrm{JHC},q}\\ \omega\end{matrix}\right)^k f(0). \tag{111}$$

The formula (109) can be inverted as follows

Theorem 2.25. *A q-analogue of* [61, p. 188, (5.50)], [89, p. 136, (3)], [94, 2.5.2].

$$\triangle_{\mathrm{JHC},q}^{n} f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x \boxplus_q \tilde{k}_q). \tag{112}$$

We will use the following abbreviations, $k_l \in \mathbb{N}$.

$$\Omega \equiv (k_1\omega_1 \boxplus_q k_2\omega_2 \boxplus_q \ldots \boxplus_q k_n\omega_n), \; k \equiv \sum_{l=1}^{n} k_l. \tag{113}$$

$$\Phi \equiv (k_1 m_1 \boxplus_q k_2 m_2 \boxplus_q \ldots \boxplus_q k_n m_n). \tag{114}$$

The notation $\sum_{\vec{k}}$ denotes a multiple summation with each of the indices $k_1, \ldots, k_n$ running between $0, 1$.

The formula (112) can be generalized to

Definition 32. Two q-analogues of [102, p. 4]

$$\underset{\omega_1,\ldots,\omega_n}{\triangle_{\mathrm{JHC},q}^{n}} f(x) \equiv (\omega_1 \ldots \omega_n)^{-1} \sum_{\vec{k}} (-1)^{n-k} f(x \boxplus_q \Omega). \tag{115}$$

$$\underset{\tilde{m}_{1q},\ldots,\tilde{m}_{nq}}{\triangle_{\mathrm{JHC},q}^{n}} f(x) \equiv (m_1 \ldots m_n)^{-1} \sum_{\vec{k}} (-1)^{n-k} f(x \boxplus_q \Phi). \tag{116}$$

There is a similar formula for $\nabla_{\mathrm{JHC},q}$.

$$\nabla_{\mathrm{JHC},q}^{n} f(x) \equiv 2^{-n} \sum_{k=0}^{n} \binom{n}{k} f(x \boxplus_q \tilde{k}_q). \tag{117}$$

In a similar way, the formula (117) can be generalized to

Definition 33. A q-analogue of [102, p. 4]

$$\underset{\omega_1,\ldots,\omega_n}{\nabla_{\mathrm{JHC},q}^{n}} f(x) \equiv 2^{-n} \sum_{\vec{k}} f(x \boxplus_q \Omega). \tag{118}$$

Theorem 2.26. *A q-analogue of* [81, (12), p. 114].

$$\nabla_{\mathrm{JHC},q}^{-1} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m} \triangle_{\mathrm{JHC},q}^{m}. \tag{119}$$

3. q-Appell polynomials

We will now describe the q-Appell polynomials, which already have been characterized by Al-Salam [6], who described its algebraic structure. In the spirit of Milne-Thomson [94, p. 125-147], which we will follow closely, we will call these q-polynomials Φ_q polynomials, and express them by a certain generating function. Examples of q-Appell polynomials or Φ_q polynomials are $B^{(n)}_{\mathrm{NWA},\nu,q}(x)$ and $E^{(n)}_{\mathrm{NWA},\nu,q}(x)$.

Definition 34. A q-analogue of [94, p. 124]. For every power series $f_n(t)$, the Φ_q polynomials of degree ν and order n have the following generating function

$$f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\Phi^{(n)}_{\nu,q}(x). \tag{120}$$

By putting $x = 0$, we have

$$f_n(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\Phi^{(n)}_{\nu,q}, \tag{121}$$

where $\Phi^{(n)}_{\nu,q}$ is called a Φ_q number of degree ν and order n.

It will be convenient to fix the value for $n = 0$ and $n = 1$.

$$\Phi^{(0)}_{\nu,q}(x) \equiv x^\nu;\ \Phi^{(1)}_{\nu,q}(x) \equiv \Phi_{\nu,q}(x). \tag{122}$$

The special case $\Phi^{(n)}_{\nu,q}(x)$ independent of x in (120) is called Eulerian generating function in [50, p. 69], [92, p. 116].

By (120) we obtain

Theorem 3.1. *A q-analogue of* [9], [94, p. 125 (4), (5)]

$$D_q\Phi^{(n)}_{\nu,q}(x) = \{\nu\}_q\Phi^{(n)}_{\nu-1,q}(x). \tag{123}$$

$$\int_a^x \Phi^{(n)}_{\nu,q}(t)\, d_q(t) = \frac{\Phi^{(n)}_{\nu+1,q}(x) - \Phi^{(n)}_{\nu+1,q}(a)}{\{\nu+1\}_q}. \tag{124}$$

By (59), (61) we obtain the two q-Taylor formulas

Theorem 3.2.

$$\Phi^{(n)}_{\nu,q}(x \oplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(n)}_{\nu-k,q}(x)y^k. \tag{125}$$

$$\Phi^{(n)}_{\nu,q}(x \boxplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{\binom{k}{2}}\Phi^{(n)}_{\nu-k,q}(x)y^k. \tag{126}$$

Note the slight difference to polynomials of q-binomial type in (125). The first formula (or [6, p. 33 2.5]) gives the symbolic equality

Theorem 3.3. *A q-analogue of* [94, p. 125 (3)]

$$\Phi^{(n)}_{\nu,q}(x) = (\Phi^{(n)}_{q} \oplus_q x)^{\nu}. \tag{127}$$

Theorem 3.4. *A q-analogue of* [94, p. 125]

$$(\mathrm{E}_q(t) - 1) f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \triangle_{\mathrm{NWA},q}\Phi^{(n)}_{\nu,q}(x). \tag{128}$$

Proof. Operate on (120) with $\triangle_{\mathrm{NWA},q}$. □

Theorem 3.5. *A q-analogue of* [94, p. 125]

$$\frac{(\mathrm{E}_q(t) + 1)}{2} f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \nabla_{\mathrm{NWA},q}\Phi^{(n)}_{\nu,q}(x). \tag{129}$$

Proof. Operate on (120) with $\nabla_{\mathrm{NWA},q}$. □

Theorem 3.6. *A q-analogue of* [94, p. 125]

$$(\mathrm{E}_{\frac{1}{q}}(t) - 1) f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \triangle_{\mathrm{JHC},q}\Phi^{(n)}_{\nu,q}(x). \tag{130}$$

Proof. Operate on (120) with $\triangle_{\mathrm{JHC},q}$. □

Theorem 3.7. *A q-analogue of* [94, p. 125]

$$\frac{(\mathrm{E}_{\frac{1}{q}}(t) + 1)}{2} f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \nabla_{\mathrm{JHC},q}\Phi^{(n)}_{\nu,q}(x). \tag{131}$$

Proof. Operate on (120) with $\nabla_{\mathrm{JHC},q}$. □

The simplest example of a Φ_q polynomial is the Rogers-Szegö polynomials [110, 131], [25, p. 90 (11)]

$$H_{n,q}(x, a) \equiv (x \oplus_q a)^n. \tag{132}$$

A special case of the Φ_q polynomials are the β_q polynomials of degree ν and order n, which are obtained by putting $f_n(t) = \frac{t^n g(t)}{(\mathrm{E}_q(t)-1)^n}$ in (120).

Definition 35.

$$\frac{t^n g(t)}{(\mathrm{E}_q(t) - 1)^n}\mathrm{E}_q(xt) \equiv \sum_{\nu=0}^{\infty} \frac{t^{\nu}\beta^{(n)}_{\nu,q}(x)}{\{\nu\}_q!}. \tag{133}$$

Theorem 3.8. [5, p. 255 10.8], *a q-analogue of* [94, (2), p. 126], [90, p. 21], [113, p. 704], [89, p. 240].

$$\triangle_{\mathrm{NWA},q}\beta_{\nu,q}^{(n)}(x) = \{\nu\}_q\beta_{\nu-1,q}^{(n-1)}(x) = D_q\beta_{\nu,q}^{(n-1)}(x). \tag{134}$$

Proof. Use (128). □

Example 3.

$$\{\nu+1\}_q\int_0^1 \beta_{\nu,q}(t)\,d_q(t) = \triangle_{\mathrm{NWA},q}\beta_{\nu+1,q}^{(1)}(0) = \delta_{0,\nu}. \tag{135}$$

By (127) the following symbolic relations obtain.

Theorem 3.9. *A q-analogue of* [94, p. 126]. *The second equation implies (26).*

$$(\beta_q^{(n)}\oplus_q x\oplus_q 1)^\nu - (\beta_q^{(n)}\oplus_q x)^\nu \doteq \{\nu\}_q(\beta_q^{(n-1)}\oplus_q x)^{\nu-1}. \tag{136}$$

$$(\beta_q^{(n)}\oplus_q 1)^\nu - \beta_{\nu,q}^{(n)} \doteq \{\nu\}_q\beta_{\nu-1,q}^{(n-1)}. \tag{137}$$

Theorem 3.10. *A q-analogue of* [101, (20), p. 163].

$$\triangle_{\mathrm{NWA},q}f(\beta_{\nu,q}^{(n)}(x)) \equiv f(\beta_{\nu,q}^{(n)}(x)\oplus_q 1) - f(\beta_{\nu,q}^{(n)}(x)) = D_q f(\beta_{\nu,q}^{(n-1)}(x)). \tag{138}$$

Theorem 3.11. *Almost a q-analogue of* [128, p. 378, (26)].

$$\sum_{k=1}^{\nu}\binom{\nu}{k}_q\beta_{\nu-k,q}^{(n)}(x) = \{\nu\}_q\beta_{\nu-1,q}^{(n-1)}(x). \tag{139}$$

Proof. Use (125) and (136). □

A special case of β_q polynomials are the generalized q-Bernoulli polynomials $B_{\mathrm{NWA},\nu,q}^{(n)}(x)$ of degree ν and order n, which were defined for $q = 1$ in [94, p. 127], [101] and for complex order in [5, p. 254, 10.3].

Definition 36. [5, p. 254, 10.3], [102, (36) p. 132], [128].The generating function for $B_{\mathrm{NWA},\nu,q}^{(n)}(x)$ is a q-analogue of [113, p. 704], [105, p. 1225, ii].

$$\frac{t^n}{(\mathrm{E}_q(t)-1)^n}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty}\frac{t^\nu B_{NWA,\nu,q}^{(n)}(x)}{\{\nu\}_q!},\ |t| < 2\pi. \tag{140}$$

This can be generalized to

Definition 37. The generating function for $B^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is the following q-analogue of [102, (77) p. 143].

$$\frac{t^n\omega_1\ldots\omega_n}{\prod_{k=1}^n(\mathrm{E}_q(\omega_k t)-1)}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu B^{(n)}_{NWA,\nu,q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!},\\ |t|<\min(|\frac{2\pi}{\omega_1}|,\ldots,|\frac{2\pi}{\omega_n}|). \tag{141}$$

Remark 7. The values for t above are for $q=1$. For general t and q, the convergence area can be different. The above definitions are mostly formal.

In the literature there are many different definitions of Bernoulli pol. One example is [81], which has an extra $n!$ in the denominator.

Obviously, $B^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$B^{(1)}_{\mathrm{NWA},\nu,q}(x|\omega)=\omega^\nu B_{\mathrm{NWA},\nu,q}(\frac{x}{\omega}). \tag{142}$$

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\mathrm{NWA},q}}\ B^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=\{\nu\}_q x^{\nu-1}. \tag{143}$$

Theorem 3.12. *The successive differences of q-Bernoulli polynomials can be expressed as q-Bernoulli polynomials. A q-analogue of* [102, (46) p. 131].

$$\underset{\omega_1,\ldots,\omega_p}{\triangle^p_{\mathrm{NWA},q}}\ B^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=\frac{\{\nu\}_q!}{\{\nu-p\}_q!}B^{(n-p)}_{\mathrm{NWA},\nu-p,q}(x|\omega_{p+1},\ldots,\omega_n), \tag{144}$$

$$B^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=(B^{(n)}_{\mathrm{NWA},\nu,\omega_1,\ldots,\omega_n,q}\oplus_q x)^\nu. \tag{145}$$

Theorem 3.13. *An explicit formula for generalized q-Bernoulli polynomials.*

$$B^{(n)}_{\mathrm{NWA},\nu,q}(x)=\sum_{k=0}^{\nu}\binom{\nu}{k}_q x^k D^{\nu-k}_{q,t}\left(\frac{t^n}{(\mathrm{E}_q(t)-1)^n}\right)|_{t=0}. \tag{146}$$

Proof. Operate with $D^\nu_{q,t}$ on both sides of (140), use the q-Leibniz theorem and, finally put $t=0$. □

The following special case is often used.

Definition 38. The Ward q-Bernoulli numbers [141, p. 265, 16.4], [5, p. 244, 4.1] are given by

$$B_{\mathrm{NWA},n,q}\equiv B^{(1)}_{\mathrm{NWA},n,q}. \tag{147}$$

The following table lists some of the first Ward q-Bernoulli numbers.

$n=0$	$n=1$	$n=2$	$n=3$
1	$-(1+q)^{-1}$	$q^2(\{3\}_q!)^{-1}$	$(1-q)q^3(\{2\}_q)^{-1}(\{4\}_q)^{-1}$

$n=4$
$q^4(1-q^2-2q^3-q^4+q^6)(\{2\}_q^2\{3\}_q\{5\}_q)^{-1}$

Theorem 3.14. *We have the following operational representation, a q-analogue of* [62, (4), p. 147], [130].

$$B^{(n)}_{\text{NWA},\nu,q}(\omega_1,\ldots,\omega_n)\ddot{=}(\oplus_{q,l=1}^{n}\omega_l B_{\text{NWA},l,q})^{\nu}. \tag{148}$$

Corollary 3.15. *A q-analogue of* [87, p. 639]. *Compare* [129, p. 192].

$$\mathrm{E}_q(tB_{\text{NWA},q})\ddot{=}\frac{t}{\mathrm{E}_q(t)-1}. \tag{149}$$

The following operator will be useful in connection with $B^{(n)}_{\text{NWA},\nu,q}(x)$.

Definition 39. Compare [28, p. 32] ($n=1$). The invertible operator $S^n_{\text{B,N},q}\in\mathbb{C}(D_q)$ is given by

$$S^n_{\text{B,N},q}\equiv\frac{(\mathrm{E}_q(D_q)-I)^n}{D_q^n}. \tag{150}$$

This implies

Theorem 3.16.

$$\triangle^n_{\text{NWA},q}=D_q^n S^n_{\text{B,N},q}. \tag{151}$$

Theorem 3.17. *A q-analogue of* [105, p. 1225, i]. *The q-Bernoulli polynomials of degree ν and order n can be expressed as*

$$B^{(n)}_{\text{NWA},\nu,q}(t)=S^{-n}_{\text{B,N},q}t^{\nu}. \tag{152}$$

Proof.

$$LHS=\sum_{k=0}^{\nu}\binom{\nu}{k}_q B^{(n)}_{\text{NWA},k,q}t^{\nu-k}=\sum_{k=0}^{\infty}\frac{B^{(n)}_{\text{NWA},k,q}}{\{k\}_q!}D_q^k t^{\nu}\overset{\text{by}(140)}{=}RHS. \tag{153}$$

□

Theorem 3.18. *A q-analogue of a generalization of* [28, p. 43, 3.3]

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B^{(n)}_{\text{NWA},\nu,q}(x\oplus_q \overline{k}_q)=\{\nu-n+1\}_{n,q}x^{\nu-n}. \tag{154}$$

Proof.

$$\triangle^n_{\text{NWA},q}B^{(n)}_{\text{NWA},\nu,q}(x)=D_q^n S^n_{\text{B,N},q}B^{(n)}_{\text{NWA},\nu,q}(x)=D_q^n S^n_{\text{B,N},q}S^{-n}_{\text{B,N},q}x^{\nu}= D_q^n x^{\nu}=\{\nu-n+1\}_{n,q}x^{\nu-n}. \tag{155}$$

□

Corollary 3.19.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B^{(n)}_{\text{NWA},\nu,q}(\overline{k}_q)=\{n\}_q!\delta_{0,\nu-n}. \tag{156}$$

Proof. Put $x=0$ in (154). □

Theorem 3.20. [5, p. 253, 9.5], *a q-analogue of* [89, (1), p. 240], [113, p. 699].

$$f(x\oplus_q B_{\text{NWA},q}\oplus_q 1)-f(x\oplus_q B_{\text{NWA},q})\doteq D_q f(x), \tag{157}$$

where here and in the sequel, we have abbreviated the umbral symbol by $B_{\text{NWA},q}$.

We will also state the corresponding equation for $B^{(n)}_{\text{NWA},\nu,q}$ written in two different forms.

Theorem 3.21. *A q-analogue of* [62, (7), p. 152], [101, (11) p. 124], [102, (36) p. 132].

$$f(x\oplus_q B^{(n)}_{\text{NWA},q}\oplus_q 1)-f(x\oplus_q B^{(n)}_{\text{NWA},q})\doteq D_q f(x\oplus_q B^{(n-1)}_{\text{NWA},q}). \tag{158}$$

$$f(B^{(n)}_{\text{NWA},q}(x)\oplus_q 1)-f(B^{(n)}_{\text{NWA},q}(x))\doteq D_q f(B^{(n-1)}_{\text{NWA},q}(x)). \tag{159}$$

Theorem 3.22. *Compare* [28, 3.15 p. 51], *where the corresponding formula for Euler polynomials was given.*

$$B_{\text{NWA},\nu,q}(x)\equiv\frac{\{\nu\}_q}{\mathrm{E}_q(D_q)-I}x^{\nu-1}=\frac{\{\nu\}_q}{E(\oplus_q)-I}x^{\nu-1}\doteq(x\oplus_q B_{\text{NWA},q})^{\nu}. \tag{160}$$

We will now follow Cigler [28] and give a few q-analogues of equations for second Bernoulli polynomials. The first two of these equations are well-known in the literature ($q=1$).

Definition 40. A q-analogue of [70, p. 87], [28, p. 13], [138, p. 575].

$$s_{\mathrm{NWA},m,q}(n) \equiv \sum_{k=0}^{n-1} (\overline{k_q})^m, \;\; s_{\mathrm{NWA},0,q}(1) \equiv 1. \tag{161}$$

Theorem 3.23. [5, p. 248, 5.13], [141, p. 265, 16.5], *a q-analogue of* [28, p. 13, p. 17: 1.11, p. 36], [89, p. 237].

$$\begin{aligned} s_{\mathrm{NWA},m,q}(n) &= \frac{B_{\mathrm{NWA},m+1,q}(\overline{(n)}_q) - B_{\mathrm{NWA},m+1,q}}{\{m+1\}_q} \\ &\equiv \frac{1}{\{m+1\}_q} \sum_{k=1}^{m+1} \binom{m+1}{k}_q (\overline{n_q})^k B_{\mathrm{NWA},m+1-k,q} \\ &\equiv \frac{1}{\{m+1\}_q} \sum_{k=0}^{m} \binom{m+1}{k}_q (\overline{n_q})^{m+1-k} B_{\mathrm{NWA},k,q}. \end{aligned} \tag{162}$$

Theorem 3.24. *A q-analogue of* [28, p. 45], [101, p. 127, (17)].

$$x^n = \int_x^{x \oplus_q 1} B_{\mathrm{NWA},n,q}(t)\, d_q(t) = \frac{B_{\mathrm{NWA},n+1,q}(x \oplus_q 1) - B_{\mathrm{NWA},n+1,q}(x)}{\{n+1\}_q}. \tag{163}$$

Proof. q-Integrate (123) for $n = 1$ and use (134). ☐

This can be rewritten as a q-analogue of the well-known identity [59, p. 496, 8.2].

$$x^n = \frac{1}{\{n+1\}_q} \sum_{k=0}^{n} \binom{n+1}{k}_q B_{\mathrm{NWA},k,q}(x). \tag{164}$$

Cigler has given some examples of translation invariant operators. One of them is the Bernoulli operator.

Definition 41. The first q-Bernoulli operator is given by the following q-integral, a q-analogue of [28, p. 91], [33, p. 154], [112, p. 59], [113, p. 701, 703], [105, p. 1217].

$$J_{\mathrm{B,N},q} f(x) \equiv \int_x^{x \oplus_q 1} f(t)\, d_q(t). \tag{165}$$

Theorem 3.25. *A q-analogue of* [28, p. 44-45], [105, p. 1217].

The first q-Bernoulli operator can be expressed in the following form.

$$J_{\mathrm{B,N},q} f(x) = \frac{\triangle_{\mathrm{NWA},q}}{D_q} f(x). \tag{166}$$

Proof. Use (152) and (163). ☐

Theorem 3.26. *A q-analogue of* [28, p. 44-45]. *We can expand a given formal power series in terms of the* $B_{\text{NWA},k,q}(x)$ *as follows.*

$$f(x)=\sum_{k=0}^{\infty}\int_{\overline{0}_q}^{\overline{1}_q} D_q^k f(t)\,d_q(t)\frac{B_{\text{NWA},k,q}(x)}{\{k\}_q!}. \tag{167}$$

Proof. Assume that

$$f(x)=\sum_{k=0}^{\infty}\frac{a_k}{\{k\}_q!}B_{\text{NWA},k,q}(x). \tag{168}$$

As we have

$$x^k=S_{\text{B,N},q}B_{\text{NWA},k,q}(x), \tag{169}$$

$$f(x)=\sum_{k=0}^{\infty}\frac{a_k}{\{k\}_q!}S_{\text{B,N},q}^{-1}x^k \tag{170}$$

$$S_{\text{B,N},q}f(x)=\sum_{k=0}^{\infty}\frac{a_k}{\{k\}_q!}x^k. \tag{171}$$

This implies

$$a_k=D_q^kS_{\text{B,N},q}f(x)|_{x=0}=D_q^k\frac{\triangle_{\text{NWA},q}}{D_q}f(x)|_{x=0}=\int_{\overline{0}_q}^{\overline{1}_q} D_q^kf(t)\,d_q(t). \tag{172}$$

□

A special case of the Φ_q polynomials are the γ_q polynomials of degree ν and order n, which are obtained by putting $f_n(t)=\frac{t^ng(t)}{(\mathrm{E}_{\frac{1}{q}}(t)-1)^n}$ in (120).

Definition 42.

$$\frac{t^ng(t)}{(\mathrm{E}_{\frac{1}{q}}(t)-1)^n}\mathrm{E}_q(xt)\equiv\sum_{\nu=0}^{\infty}\frac{t^\nu\gamma_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{173}$$

Theorem 3.27. *A q-analogue of* [94, (2), p. 126], [90, p. 21], [113, p. 704].

$$\triangle_{\text{JHC},q}\gamma_{\nu,q}^{(n)}(x)=\{\nu\}_q\gamma_{\nu-1,q}^{(n-1)}(x)=D_q\gamma_{\nu,q}^{(n-1)}(x). \tag{174}$$

Proof. Use (130). □

By (131) the following symbolic relations obtain.

Theorem 3.28. *A q-analogue of* [94, p. 126]. *The second equation implies (177).*

$$(\gamma_q^{(n)}\oplus_q x\boxplus_q 1)^\nu-(\gamma_q^{(n)}\oplus_q x)^\nu=\{\nu\}_q(\gamma_q^{(n-1)}\oplus_q x)^{\nu-1}. \tag{175}$$

$$(\gamma_q^{(n)}\boxplus_q 1)^\nu-\gamma_{\nu,q}^{(n)}=\{\nu\}_q\gamma_{\nu-1,q}^{(n-1)}. \tag{176}$$

$$B_{\mathrm{JHC},0,q}=1,\ (B_{\mathrm{JHC},q}\boxplus_q\ 1)^k-B_{\mathrm{JHC},k,q}=\delta_{1,k}. \tag{177}$$

where $B_{\mathrm{JHC},q}^n$ is replaced by $B_{\mathrm{JHC},n,q}$ on expansion.

Theorem 3.29. *A q-analogue of* [101, (20), p. 163].

$$\triangle_{\mathrm{JHC},q}f(\gamma_{\nu,q}^{(n)}(x))\equiv f(\gamma_{\nu,q}^{(n)}(x)\boxplus_q 1)-f(\gamma_{\nu,q}^{(n)}(x))=D_q f(\gamma_{\nu,q}^{(n-1)}(x)). \tag{178}$$

Theorem 3.30. *Almost a q-analogue of* [128, p. 378, (26)].

$$\sum_{k=1}^{\nu}\binom{\nu}{k}_q\gamma_{\nu-k,q}^{(n)}(x)q^{\binom{k}{2}}=\{\nu\}_q\gamma_{\nu-1,q}^{(n-1)}(x). \tag{179}$$

Proof. Use (125) and (136). □

A special case of γ_q polynomials are the second generalized q-Bernoulli polynomials $B_{\mathrm{JHC},\nu,q}^{(n)}(x)$ of degree ν and order n, which were defined for $q=1$ in [94, p. 127], [101].

Definition 43. The generating function for $B_{\mathrm{JHC},\nu,q}^{(n)}(x)$ is a q-analogue of [113, p. 704], [105, p. 1225, ii].

$$\frac{t^n}{(\mathrm{E}_{\frac{1}{q}}(t)-1)^n}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu B_{JHC,\nu,q}^{(n)}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{180}$$

This can be generalized to

Definition 44. The generating function for $B_{\mathrm{JHC},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)$ is the following q-analogue of [102, (77) p. 143].

$$\begin{aligned}&\frac{t^n\omega_1\ldots\omega_n}{\prod_{k=1}^n(\mathrm{E}_{\frac{1}{q}}(\omega_k t)-1)}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu B_{JHC,\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!},\\&|t|<\min(|\frac{2\pi}{\omega_1}|,\ldots,|\frac{2\pi}{\omega_n}|).\end{aligned} \tag{181}$$

Obviously, $B_{\mathrm{JHC},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$B_{\mathrm{JHC},\nu,q}^{(1)}(x|\omega)=\omega^\nu B_{\mathrm{JHC},\nu,q}(\frac{x}{\omega}). \tag{182}$$

$$\underset{\omega_1,\ldots,\omega_n}{\triangle_{\mathrm{JHC},q}^n}B_{\mathrm{JHC},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)=\{\nu\}_q x^{\nu-1}. \tag{183}$$

Theorem 3.31. *The successive differences of the second q-Bernoulli polynomials can be expressed as q-Bernoulli polynomials. A q-analogue of* [102, (46) p. 131].

$$\underset{\omega_1,\ldots,\omega_p}{\triangle^p_{\text{JHC},q}} B^{(n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n) = \frac{\{\nu\}_q!}{\{\nu-p\}_q!} B^{(n-p)}_{\text{JHC},\nu-p,q}(x|\omega_{p+1},\ldots,\omega_n), \tag{184}$$

$$B^{(n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n) = (B^{(n)}_{\text{JHC},\nu,\omega_1,\ldots,\omega_n,q} \oplus_q x)^\nu. \tag{185}$$

Theorem 3.32. *An explicit formula for the second generalized q-Bernoulli polynomials.*

$$B^{(n)}_{\text{JHC},\nu,q}(x) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q x^k D^{\nu-k}_{q,t} \left(\frac{t^n}{(\mathrm{E}_{\frac{1}{q}}(t)-1)^n} \right) |_{t=0}. \tag{186}$$

Proof. Operate with $D^\nu_{q,t}$ on both sides of (180), use the q-Leibniz theorem and, finally put $t = 0$. □

The following special case is often used.

Definition 45. The Jackson q-Bernoulli numbers [141, p. 265, 16.4], [5, p. 244, 4.1] are given by

$$B_{\text{JHC},n,q} \equiv B^{(1)}_{\text{JHC},n,q}. \tag{187}$$

The following table lists some of the smallest Jackson q-Bernoulli numbers.

$n=0$	$n=1$	$n=2$	$n=3$
1	$-q(1+q)^{-1}$	$q^2(\{3\}_q!)^{-1}$	$(q^4-q^3)(\{2\}_q)^{-1}(\{4\}_q)^{-1}$

$n=4$
$q^4(1-q^2-2q^3-q^4+q^6)(\{2\}_q^2\{3\}_q\{5\}_q)^{-1}$

The astute reader has certainly noticed a certain resemblance between the two q-Bernoulli numbers. By an elementary argument for the generating function we can prove the following symmetry theorem:

Theorem 3.33. *For ν even,*

$$B_{\text{NWA},\nu,q} = B_{\text{JHC},\nu,q}. \tag{188}$$

For ν odd,

$$B_{\text{NWA},\nu,q} = -B_{\text{JHC},\nu,q}. \tag{189}$$

Theorem 3.34. *We have the following operational representation, a q-analogue of* [130].

$$B^{(n)}_{\mathrm{JHC},\nu,q}(\omega_1,\ldots,\omega_n)\ddot{=}(\oplus^n_{q,l=1}\omega_l B_{\mathrm{JHC},l,q})^\nu. \tag{190}$$

Corollary 3.35. *Compare* [129, p. 192].

$$\mathrm{E}_{\frac{1}{q}}(tB_{\mathrm{JHC},q})\ddot{=}\frac{t}{\mathrm{E}_{\frac{1}{q}}(t)-1}. \tag{191}$$

The following operator will be useful in connection with $B^{(n)}_{\mathrm{JHC},\nu,q}(x)$.

Definition 46. Compare [28, p. 32] ($n=1$). The invertible operator $S^n_{\mathrm{B,J},q}\in\mathbb{C}(D_q)$ is given by

$$S^n_{\mathrm{B,J},q}\equiv\frac{(\mathrm{E}_{\frac{1}{q}}(D_q)-I)^n}{D^n_q}. \tag{192}$$

This implies

Theorem 3.36.

$$\triangle^n_{\mathrm{JHC},q}=D^n_q S^n_{\mathrm{B,J},q}. \tag{193}$$

Theorem 3.37. *A q-analogue of* [105, p. 1225, i]. *The second q-Bernoulli polynomials of degree ν and order n can be expressed as*

$$B^{(n)}_{\mathrm{JHC},\nu,q}(t)=S^{-n}_{\mathrm{B,J},q}t^\nu. \tag{194}$$

Proof.

$$LHS=\sum_{k=0}^{\nu}\binom{\nu}{k}_q B^{(n)}_{\mathrm{JHC},k,q}t^{\nu-k}=\sum_{k=0}^{\infty}\frac{B^{(n)}_{\mathrm{JHC},k,q}}{\{k\}_q!}D^k_q t^\nu\overset{\text{by}(180)}{=}RHS. \tag{195}$$

□

Theorem 3.38. *A q-analogue of a generalization of* [28, p. 43, 3.3]

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B^{(n)}_{\mathrm{JHC},\nu,q}(x\boxplus_q\tilde{k}_q)=\{\nu-n+1\}_{n,q}x^{\nu-n}. \tag{196}$$

Proof.

$$\begin{aligned}&\triangle^n_{\mathrm{JHC},q}B^{(n)}_{\mathrm{JHC},\nu,q}(x)=D^n_q S^n_{\mathrm{B,J},q}B^{(n)}_{\mathrm{JHC},\nu,q}(x)=D^n_q S^n_{\mathrm{B,J},q}S^{-n}_{\mathrm{B,J},q}x^\nu=\\&D^n_q x^\nu=\{\nu-n+1\}_{n,q}x^{\nu-n}.\end{aligned} \tag{197}$$

□

Corollary 3.39.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B^{(n)}_{\mathrm{JHC},\nu,q}(\tilde{k}_q)=\{n\}_q!\delta_{0,\nu-n}. \tag{198}$$

Proof. Put $x=0$ in (196). □

Theorem 3.40. [5, p. 253, 9.5], *a q-analogue of* [89, (1), p. 240], [113, p. 699].

$$f(x\oplus_q B_{\mathrm{JHC},q}\boxplus_q 1)-f(x\oplus_q B_{\mathrm{JHC},q})\ddot{=}D_q f(x). \tag{199}$$

We will also state the corresponding equation for $B^{(n)}_{\mathrm{JHC},\nu,q}$ written in two different forms.

Theorem 3.41. *A q-analogue of* [101, (11) p. 124], [102, (36) p. 132].

$$f(x\oplus_q B^{(n)}_{\mathrm{JHC},q}\boxplus_q 1)-f(x\oplus_q B^{(n)}_{\mathrm{JHC},q})\ddot{=}D_q f(x\boxplus_q B^{(n-1)}_{\mathrm{JHC},q}). \tag{200}$$

$$f(B^{(n)}_{\mathrm{JHC},q}(x)\boxplus_q 1)-f(B^{(n)}_{\mathrm{JHC},q}(x))\ddot{=}D_q f(B^{(n-1)}_{\mathrm{JHC},q}(x)). \tag{201}$$

Theorem 3.42. *Compare* [28, 3.15 p. 51], *where the corresponding formula for Euler polynomials was given.*

$$B_{\mathrm{JHC},\nu,q}(x)\equiv\frac{\{\nu\}_q}{\mathrm{E}_{\frac{1}{q}}(D_q)-I}x^{\nu-1}=\frac{\{\nu\}_q}{E(\boxplus_q)-I}x^{\nu-1}\ddot{=}(x\oplus_q B_{\mathrm{JHC},q})^{\nu}. \tag{202}$$

We will now follow Cigler [28] and give a few q-analogues of equations for Bernoulli polynomials. The first two of these equations are well-known in the literature ($q=1$).

Definition 47. A q-analogue of [70, p. 87], [28, p. 13], [138, p. 575].

$$s_{\mathrm{JHC},m,q}(n)\equiv\sum_{k=0}^{n-1}(\tilde{k_q})^m,\ s_{\mathrm{JHC},0,q}(1)\equiv 1. \tag{203}$$

Theorem 3.43. *A q-analogue of* [28, p. 13, p. 17: 1.11, p. 36], [89, p. 237].

$$\begin{aligned} s_{\mathrm{JHC},m,q}(n)&=\frac{B_{\mathrm{JHC},m+1,q}(\widetilde{(n)}_q)-B_{\mathrm{JHC},m+1,q}}{\{m+1\}_q}\\ &\equiv\frac{1}{\{m+1\}_q}\sum_{k=1}^{m+1}\binom{m+1}{k}_q(\tilde{n_q})^k B_{\mathrm{JHC},m+1-k,q} \\ &\equiv\frac{1}{\{m+1\}_q}\sum_{k=0}^{m}\binom{m+1}{k}_q(\tilde{n_q})^{m+1-k} B_{\mathrm{JHC},k,q}.\end{aligned} \tag{204}$$

Theorem 3.44. *A q-analogue of* [28, p. 45], [101, p. 127, (17)].

$$x^n=\int_x^{x\boxplus_q 1}B_{\mathrm{JHC},n,q}(t)\,d_q(t)=\frac{B_{\mathrm{JHC},n+1,q}(x\boxplus_q 1)-B_{\mathrm{JHC},n+1,q}(x)}{\{n+1\}_q}. \tag{205}$$

Proof. q-Integrate (123) for $n = 1$ and use (174). □

This can be rewritten as a q-analogue of the well-known identity [59, p. 496, 8.2].

$$x^n = \frac{1}{\{n+1\}_q} \sum_{k=0}^{n} \binom{n+1}{k}_q B_{\text{JHC},k,q}(x) q^{\binom{n+1-k}{2}}. \tag{206}$$

Cigler has given some examples of translation invariant operators. One of them is the Bernoulli operator.

Definition 48. The second q-Bernoulli operator is given by the following q-integral, a q-analogue of [28, p. 91], [33, p. 154], [112, p. 59], [113, p. 701, 703], [105, p. 1217].

$$J_{\text{B,J},q} f(x) \equiv \int_{x}^{x \boxplus_q 1} f(t)\, d_q(t). \tag{207}$$

Theorem 3.45. *A q-analogue of* [28, p. 44-45], [105, p. 1217].

The second q-Bernoulli operator can be expressed in the following form.

$$J_{\text{B,J},q} f(x) = \frac{\triangle_{\text{JHC},q}}{D_q} f(x). \tag{208}$$

Proof. Use (194) and (205). □

Theorem 3.46. *A q-analogue of* [28, p. 44-45]. *We can expand a given formal power series in terms of the $B_{\text{JHC},k,q}(x)$ as follows.*

$$f(x) = \sum_{k=0}^{\infty} \int_{\tilde{0}_q}^{\tilde{1}_q} D_q^k f(t)\, d_q(t) \frac{B_{\text{JHC},k,q}(x)}{\{k\}_q!}. \tag{209}$$

Proof. Same as for NWA. □

A special case of the Φ_q polynomials are the η_q polynomials of order n, which are obtained by putting $f_n(t) = \frac{g(t)2^n}{(\text{E}_q(t)+1)^n}$ in (120).

Definition 49. A q-analogue of [94, p. 142, (1)].

$$\frac{2^n}{(\text{E}_q(t)+1)^n} g(t) \text{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \eta_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{210}$$

By (129) we get a q-analogue of [94], [93, p. 519].

$$\nabla_{\text{NWA},q} \eta_{\nu,q}^{(n)}(x) = \eta_{\nu,q}^{(n-1)}(x). \tag{211}$$

We will now define the first q-Euler polynomials, a special case of the η_q polynomials. There are many similar definitions of these, but we will follow [94, p. 143-147], [28, p. 51], because it is equivalent to the q-Appell polynomials from [6].

Definition 50. The generating function for the first q-Euler polynomials of degree ν and order n $E^{(n)}_{\text{NWA},\nu,q}(x)$ is the following q-analogue of [111, p. 102], [94, p. 309], [133, p. 345].

$$\frac{2^n \mathrm{E}_q(xt)}{(\mathrm{E}_q(t)+1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} E^{(n)}_{\text{NWA},\nu,q}(x), \ |t| < \pi. \tag{212}$$

This can be generalized to

Definition 51. The generating function for the first q-Euler polynomials of degree ν and order n $E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is the following q-analogue of [102, p. 143 (78)].

$$\begin{aligned}&\frac{2^n \mathrm{E}_q(xt)}{\prod_{k=1}^n(\mathrm{E}_q(\omega_k t)+1)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n),\\ &|t| < \min(|\frac{\pi}{\omega_1}|,\ldots,|\frac{\pi}{\omega_n}|).\end{aligned} \tag{213}$$

Obviously, $E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$E^{(1)}_{\text{NWA},\nu,q}(x|\omega) = \omega^\nu E_{\text{NWA},\nu,q}(\frac{x}{\omega}). \tag{214}$$

From

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}} \ E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n) = x^\nu \tag{215}$$

we obtain

$$\underset{\omega_1,\ldots,\omega_p}{\nabla^p_{\text{NWA},q}} \ E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n) = E^{(n-p)}_{\text{NWA},\nu,q}(x|\omega_{p+1},\ldots,\omega_n). \tag{216}$$

Theorem 3.47. *A q-analogue of* [94, p. 144, (7)], [102, (7), p.121], [128, p. 378, (28)].

$$\sum_{k=0}^{\nu} \binom{\nu}{k}_q E^{(n)}_{\text{NWA},\nu-k,q}(x) + E^{(n)}_{\text{NWA},\nu,q}(x) = 2E^{(n-1)}_{\text{NWA},\nu,q}(x). \tag{217}$$

With this formula we can compute all first q-Euler polynomials of order n, given knowledge of the polynomials of order $n-1$.

Definition 52. A q-analogue of [101, p. 139], [89, p. 252]. The first generalized q-Euler numbers are given by

$$F^{(n)}_{\text{NWA},\nu,q} \equiv E^{(n)}_{\text{NWA},\nu,q}(0). \tag{218}$$

Furthermore we put

$$F_{\mathrm{NWA},k,q} \equiv F^{(1)}_{\mathrm{NWA},k,q};\ E_{\mathrm{NWA},\nu,q}(x) \equiv E^{(1)}_{\mathrm{NWA},\nu,q}(x). \tag{219}$$

Remark 8. The numbers $F_{\mathrm{NWA},k,q}$ are q-analogues of the numbers in [93, p. 520], [28, p. 51], which are multiples of the tangent numbers [115, p. 296]. Lucas [89, p. 250] called them G after Genocchi, but I disagree with this.

The q-analogues of the original integral Euler numbers (secant numbers), see Salié [115], appear in [8].

Remark 9. When $q = 1$, our $E_\nu(x) = \nu! E_{J,\nu}(x)$, where $E_{J,\nu}(x)$ is the Euler polynomial used in [81].

Theorem 3.48. *The operator expression is a q-analogue of* [28, 3.15 p. 51].

$$E_{\mathrm{NWA},\nu,q}(x) \equiv \frac{2}{\mathrm{E}_q(D_q) + I} x^\nu = \frac{2}{E(\oplus_q) + I} x^\nu \doteq (x \oplus_q F_{\mathrm{NWA},q})^\nu. \tag{220}$$

The following 2 recursion formulas are quite useful for the computations of the first q-Euler pol.

Theorem 3.49. *A q-analogue of* [102, (27), p. 24], [128, p. 378, (29)].

$$E_{\mathrm{NWA},\nu,q}(x) + \sum_{k=0}^{\nu} \binom{\nu}{k}_q E_{\mathrm{NWA},k,q}(x) = 2x^\nu. \tag{221}$$

Theorem 3.50. *A q-analogue of* [28, 3.16 p. 51], [89, p. 252].

$$(1 \oplus_q F_{\mathrm{NWA},q})^n + (F_{\mathrm{NWA},q})^n \doteq 2\delta_{0,n}. \tag{222}$$

Theorem 3.51. *A q-analogue of* [18, p. 6 (4.3)], [101, (19), p. 136], *a corrected version of* [89, p. 261].

$$f(x \oplus_q F_{\mathrm{NWA},q} \oplus_q 1) + f(x \oplus_q F_{\mathrm{NWA},q}) \doteq 2f(x). \tag{223}$$

We will also state the corresponding equation for $E^{(n)}_{\mathrm{NWA},\nu,q}$ written in two different forms.

Theorem 3.52. *A q-analogue of* [101, (19), p. 150, p. 155], [102, (29) p. 126].

$$\begin{aligned}&\nabla_{\mathrm{NWA},q} f(x \oplus_q F^{(n)}_{\mathrm{NWA},q}) \doteq f(x \oplus_q F^{(n-1)}_{\mathrm{NWA},q}) \doteq \\ &\nabla_{\mathrm{NWA},q} f(E^{(n)}_{\mathrm{NWA},q}(x)) \doteq f(E^{(n-1)}_{\mathrm{NWA},q}(x)).\end{aligned} \tag{224}$$

The following table lists some of the first q-Euler numbers $F_{\text{NWA},n,q}$.

$n=0$	$n=1$	$n=2$	$n=3$
1	-2^{-1}	$2^{-2}(-1+q)$	$2^{-3}(-1+2q+2q^2-q^3)$

$n=4$
$2^{-4}(q-1)\{3\}_q!(q^2-4q+1)$

Theorem 3.53. *We have the following operational representation, a q-analogue of* [130].

$$E^{(n)}_{\text{NWA},\nu,q}(\omega_1,\ldots,\omega_n)\ddot{=}(\oplus_{q,l=1}^{n}\omega_l E_{\text{NWA},l,q})^{\nu}. \tag{225}$$

Corollary 3.54.

$$\mathrm{E}_q(tE_{\text{NWA},q})\ddot{=}\frac{2}{\mathrm{E}_q(t)+1}. \tag{226}$$

Theorem 3.55. *The first q-Euler pol. can be expressed as a finite sum of diff. operators on x^n. Almost a q-analogue of* [81, p. 289].

$$E_{\text{NWA},n,q}(x)=\sum_{m=0}^{n}\frac{(-1)^m}{2^m}\triangle^m_{\text{NWA},q}x^n. \tag{227}$$

Theorem 3.56. *A generalization of (221).*

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}E^{(n)}_{\text{NWA},\nu,q}(x\oplus_q\overline{k}_q)=x^{\nu}. \tag{228}$$

Proof. Develop $\nabla^n_{\text{NWA},q}E^{(n)}_{\text{NWA},\nu,q}(x)$. □

Definition 53. A q-analogue of [70, p. 88]. The notation from N. Nielsen (1865–1925) [98, p. 401] is a slightly modified variant of the original paper by Lucas [89].

$$\sigma_{\text{NWA},m,q}(n)\equiv\sum_{k=0}^{n-1}(-1)^k(\overline{k_q})^m \tag{229}$$

Theorem 3.57. *A q-analogue of* [28, p. 53], [94, p. 307], [53, p. 136].

$$\sigma_{\text{NWA},m,q}(n)=\frac{(-1)^{n-1}E_{\text{NWA},m,q}(\overline{n}_q)+E_{\text{NWA},m,q}(\overline{0_q})}{2}. \tag{230}$$

Proof.

$$\begin{aligned} LHS = \sum_{k=0}^{n-1}(-1)^k \nabla_{\text{NWA},q} E_{\text{NWA},m,q}(\overline{k_q}) = \\ \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k (E_{\text{NWA},m,q}(\overline{k_q} \oplus_q 1) + E_{\text{NWA},m,q}(\overline{k_q})) = RHS. \end{aligned} \tag{231}$$

□

A special case of the Φ_q polynomials are the θ_q polynomials of order n, which are obtained by putting $f_n(t) = \frac{g(t)2^n}{(\mathrm{E}_{\frac{1}{q}}(t)+1)^n}$ in (120).

Definition 54. A q-analogue of [94, p. 142, (1)].

$$\frac{2^n}{(\mathrm{E}_{\frac{1}{q}}(t) + 1)^n} g(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \theta_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{232}$$

By (131) we get a q-analogue of [94], [93, p. 519].

$$\nabla_{\text{JHC},q} \theta_{\nu,q}^{(n)}(x) = \theta_{\nu,q}^{(n-1)}(x). \tag{233}$$

We will now define second q-Euler polynomials, a special case of the θ_q polynomials. There are many similar definitions of these, but we will follow [94, p. 143-147], [28, p. 51], because it is equivalent to the q-Appell polynomials from [6].

Definition 55. The generating function for the second q-Euler polynomials of degree ν and order n $E_{\text{JHC},\nu,q}^{(n)}(x)$ is the following q-analogue of [111, p. 102], [94, p. 309], [133, p. 345].

$$\frac{2^n \mathrm{E}_q(xt)}{(\mathrm{E}_{\frac{1}{q}}(t) + 1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} E_{\text{JHC},\nu,q}^{(n)}(x), \ |t| < \pi. \tag{234}$$

This can be generalized to

Definition 56. The generating function for the second q-Euler polynomials of degree ν and order n $E_{\text{JHC},\nu,q}^{(n)}(x|\omega_1, \ldots, \omega_n)$ is the following q-analogue of [102, p. 143 (78)].

$$\begin{aligned} \frac{2^n \mathrm{E}_q(xt)}{\prod_{k=1}^n (\mathrm{E}_{\frac{1}{q}}(\omega_k t) + 1)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} E_{\text{JHC},\nu,q}^{(n)}(x|\omega_1, \ldots, \omega_n), \\ |t| < \min(|\frac{\pi}{\omega_1}|, \ldots, |\frac{\pi}{\omega_n}|). \end{aligned} \tag{235}$$

Obviously, $E^{(n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$E^{(1)}_{\text{JHC},\nu,q}(x|\omega)=\omega^{\nu}E_{\text{JHC},\nu,q}(\frac{x}{\omega}). \tag{236}$$

From

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^{n}_{\text{JHC},q}}\ E^{(n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n)=x^{\nu} \tag{237}$$

we obtain

$$\underset{\omega_1,\ldots,\omega_p}{\nabla^{p}_{\text{JHC},q}}\ E^{(n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n)=E^{(n-p)}_{\text{JHC},\nu,q}(x|\omega_{p+1},\ldots,\omega_n). \tag{238}$$

Theorem 3.58. *A q-analogue of* [94, p. 144, (7)], [102, (7), p.121], [128, p. 378, (28)].

$$\sum_{k=0}^{\nu}\binom{\nu}{k}_q q^{\binom{k}{2}}E^{(n)}_{\text{JHC},\nu-k,q}(x)+E^{(n)}_{\text{JHC},\nu,q}(x)=2E^{(n-1)}_{\text{JHC},\nu,q}(x). \tag{239}$$

With this formula we can compute all second q-Euler polynomials of order n, given knowledge of the polynomials of order $n-1$.

Definition 57. A q-analogue of [101, p. 139], [89, p. 252]. The second generalized q-Euler numbers are given by

$$F^{(n)}_{\text{JHC},\nu,q}\equiv E^{(n)}_{\text{JHC},\nu,q}(0). \tag{240}$$

Furthermore we put

$$F_{\text{JHC},k,q}\equiv F^{(1)}_{\text{JHC},k,q};\ E_{\text{JHC},\nu,q}(x)\equiv E^{(1)}_{\text{JHC},\nu,q}(x). \tag{241}$$

Remark 10. The numbers $F_{\text{JHC},k,q}$ are q-analogues of the numbers in [93, p. 520], [28, p. 51], which are multiples of the tangent numbers [115, p. 296].

The q-analogues of the original integral Euler numbers (secant numbers), see Salié [115], appear in [8]. It was J.-L. Raabe (1801-1859), who in 1851 first used the name Euler numbers. It was then used by Sylvester, Catalan, Glaisher, Lucas and from 1877 the name was used in Germany. Also see [70, p. 79].

Remark 11. When $q=1$, our $E_{\nu}(x)=\nu!E_{J,\nu}(x)$, where $E_{J,\nu}(x)$ is the Euler polynomial used in [81].

Theorem 3.59. *The operator expression is a q-analogue of* [28, 3.15 p. 51].

$$E_{\text{JHC},\nu,q}(x)\equiv\frac{2}{\mathrm{E}_{\frac{1}{q}}(D_q)+I}x^{\nu}=\frac{2}{E(\boxplus_q)+I}x^{\nu}\doteq(x\oplus_q F_{\text{JHC},q})^{\nu}. \tag{242}$$

The following 2 recursion formulas are quite useful for the computations of the second q-Euler pol.

Theorem 3.60. *A q-analogue of* [102, (27), p. 24], [128, p. 378, (29)].

$$E_{\mathrm{JHC},\nu,q}(x)+\sum_{k=0}^{\nu}\binom{\nu}{k}_q q^{\binom{\nu-k}{2}}E_{\mathrm{JHC},k,q}(x)=2x^{\nu}. \tag{243}$$

Theorem 3.61. *A q-analogue of* [28, 3.16 p. 51], [89, p. 252].

$$(F_{\mathrm{JHC},q\boxplus_q 1})^n+(F_{\mathrm{JHC},q})^n\ddot{=}2\delta_{0,n}. \tag{244}$$

Theorem 3.62. *A q-analogue of* [89, p. 261], [18, p. 6 (4.3)], [101, (19), p. 136].

$$f(x\boxplus_q F_{\mathrm{JHC},q}\boxplus_q 1)+f(x\oplus_q F_{\mathrm{JHC},q})\ddot{=}2f(x). \tag{245}$$

We will also state the corresponding equation for $E^{(n)}_{\mathrm{JHC},\nu,q}$ written in two different forms.

Theorem 3.63. *A q-analogue of* [101, (19), p. 150, p. 155], [102, (29) p. 126].

$$\begin{aligned}&\nabla_{\mathrm{JHC},q}f(x\oplus_q F^{(n)}_{\mathrm{JHC},q})\ddot{=}f(x\oplus_q F^{(n-1)}_{\mathrm{JHC},q})\ddot{=}\\&\nabla_{\mathrm{JHC},q}f(E^{(n)}_{\mathrm{JHC},q}(x))\ddot{=}f(E^{(n-1)}_{\mathrm{JHC},q}(x)).\end{aligned} \tag{246}$$

The following table lists some of the first $F_{\mathrm{JHC},n,q}$.

$n=0$	$n=1$	$n=2$	$n=3$
1	-2^{-1}	$2^{-2}(1-q)$	$2^{-3}(-1+2q+2q^2-q^3)$

$n=4$
$2^{-4}(1-3q-3q^2+3q^4+3q^5-q^6)$

Theorem 3.64. *We have the following operational representation.*

$$E^{(n)}_{\mathrm{JHC},\nu,q}(\omega_1,\ldots,\omega_n)\ddot{=}(\oplus_{q,l=1}^{n}\omega_l E_{\mathrm{JHC},l,q})^{\nu}. \tag{247}$$

Corollary 3.65.

$$\mathrm{E}_q(tE_{\mathrm{JHC},q})\ddot{=}\frac{2}{\mathrm{E}_{\frac{1}{q}}(t)+1}. \tag{248}$$

Theorem 3.66. *The second q-Euler pol. can be expressed as a finite sum of diff. operators on x^n. Almost a q-analogue of* [81, p. 289].

$$E_{\mathrm{JHC},n,q}(x)=\sum_{m=0}^{n}\frac{(-1)^m}{2^m}\triangle^m_{\mathrm{JHC},q}x^n. \tag{249}$$

Theorem 3.67. *A generalization of (243).*

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}E^{(n)}_{\mathrm{JHC},\nu,q}(x\boxplus_q\tilde{k}_q)=x^{\nu}. \tag{250}$$

Proof. Develop $\nabla^n_{\mathrm{JHC},q}E^{(n)}_{\mathrm{JHC},\nu,q}(x)$. ☐

Definition 58. A q-analogue of [70, p. 88].

$$\sigma_{\mathrm{JHC},m,q}(n)\equiv\sum_{k=0}^{n-1}(-1)^k(\tilde{k}_q)^m. \tag{251}$$

Theorem 3.68. *A q-analogue of* [28, p. 53], [94, p. 307], [53, p. 136].

$$\sigma_{\mathrm{JHC},m,q}(n)=\frac{(-1)^{n-1}E_{\mathrm{JHC},m,q}(\tilde{n}_q)+E_{\mathrm{JHC},m,q}(\tilde{0}_q)}{2}. \tag{252}$$

Proof.

$$\begin{aligned}&LHS=\sum_{k=0}^{n-1}(-1)^k\nabla_{\mathrm{JHC},q}E_{\mathrm{JHC},m,q}(\tilde{k}_q)=\\&\frac{1}{2}\sum_{k=0}^{n-1}(-1)^k(E_{\mathrm{JHC},m,q}(\tilde{k}_q\boxplus_q 1)+E_{\mathrm{JHC},m,q}(\tilde{k}_q))=RHS.\end{aligned} \tag{253}$$

☐

So far we considered only q-Bernoulli polynomials and q-Euler polynomials of positive order n. As the sequel shows, it will be useful to allow n also to be a negative integer. The following calculations are q-analogues of Nörlund [102, p. 133 ff]

Definition 59. As a q-analogue of [102, (50) p. 133], [105, p. 1226, xvi] and [128, p. 378, (19)], we define first q-Bernoulli polynomials of two variables as

$$\begin{aligned}&B^{(n+p)}_{\mathrm{NWA},\nu,q}(x\oplus_q y|\omega_1,\ldots,\omega_{n+p})=\\&=(B^{(n)}_{\mathrm{NWA},q}(x|\omega_1,\ldots,\omega_n)\oplus_q B^{(p)}_{\mathrm{NWA},q}(y|\omega_{n+1},\ldots,\omega_{n+p}))^{\nu},\end{aligned} \tag{254}$$

where we assume that n and p operate on x and y respectively, and the same for any $q-$polynomial.

The relation (142) together with (254) show, that $B^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is a homogeneous function of $x,\omega_1,\ldots,\omega_n$ of degree ν, a q-analogue of [102, p. 134 (55)], i.e.

$$B^{(n)}_{\mathrm{NWA},\nu,q}(\lambda x|\lambda\omega_1,\ldots,\lambda\omega_n)=\lambda^{\nu}B^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n),\ \lambda\in\mathbb{C}. \tag{255}$$

And the same for $q-$ Euler-, Lucas- and $G-$ polynomials.

This can be generalized in at least two ways

Theorem 3.69. *A q-analogue of* [113, p. 704], [102, p. 133]. *If* $\sum_{l=1}^{s} n_l = n$,

$$B_{\text{NWA},k,q}^{(n)}(x_1 \oplus_q \ldots \oplus_q x_s) = \sum_{m_1+\ldots+m_s=k} \binom{k}{m_1,\ldots,m_s}_q \prod_{j=1}^{s} B_{\text{NWA},m_j,q}^{(n_j)}(x_j). \tag{256}$$

where we assume that n_j operates on x_j. And the same for any $q-$polynomial.

Proof. In umbral notation we have, as in the classical case

$$(x_1 \oplus_q \ldots \oplus_q x_s \oplus_q \tilde{n}_q \gamma)^k \sim ((x_1 \oplus_q \tilde{n_1}_q \gamma') \oplus_q \ldots \oplus_q (x_s \oplus_q \tilde{n_s}_q \gamma''))^k, \tag{257}$$

where $\gamma', \ldots, \gamma''$ are distinct umbrae, each equivalent to γ. □

Theorem 3.70. *A q-analogue of* [113, p. 704], [102, p. 133]. *If* $\sum_{l=1}^{s} n_l = n$,

$$B_{\text{NWA},k,q}^{(n)}(x_1 \boxplus_q \ldots \boxplus_q x_s) = \sum_{m_1+\ldots+m_s=k} \binom{k}{m_1,\ldots,m_s}_q \prod_{j=1}^{s} B_{\text{NWA},m_j,q}^{(n_j)}(x_j) q^{\binom{\vec{m}}{2}}, \tag{258}$$

$\vec{m} = (m_2, \ldots, m_n)$. *We assume that n_j operates on x_j. And the same for any $q-$polynomial.*

By (134) and (211) we get

$$\triangle_{\text{NWA},q}^{n} B_{\text{NWA},\nu,q}^{(n)}(x) = \frac{\{\nu\}_q!}{\{\nu-n\}_q!} x^{\nu-n},$$
$$\nabla_{\text{NWA},q}^{n} E_{\text{NWA},\nu,q}^{(n)}(x) = x^{\nu},$$

and we have

Definition 60. A q-analogue of [101, p. 177], [102, (66), p. 138]. The first q-Bernoulli polynomials of negative order $-n$ are given by

$$B_{\text{NWA},\nu,q}^{(-n)}(x|\omega_1,\ldots,\omega_n) \equiv \frac{\{\nu\}_q!}{\{\nu+n\}_q!} \underset{\omega_1,\ldots,\omega_n}{\triangle_{\text{NWA},q}^{n}} x^{\nu+n}, \tag{259}$$

and the first q-Euler polynomial of negative order $-n$ by the following q-analogue of [102, (67) p. 138]

$$E_{\text{NWA},\nu,q}^{(-n)}(x|\omega_1,\ldots,\omega_n) \equiv \underset{\omega_1,\ldots,\omega_n}{\nabla_{\text{NWA},q}^{n}} x^{\nu}, \tag{260}$$

where $\nu, n \in \mathbb{N}$. This defines q-Bernoulli- and q-Euler polynomials of negative order as iterated $\triangle_{\text{NWA},q}$ and $\nabla_{\text{NWA},q}$ operating on positive integer powers of x.

Furthermore,

$$B^{(-n)}_{\mathrm{NWA},\nu,q} \equiv B^{(-n)}_{\mathrm{NWA},\nu,q}(0), \tag{261}$$

$$F^{(-n)}_{\mathrm{NWA},\nu,q} \equiv E^{(-n)}_{\mathrm{NWA},\nu,q}(0). \tag{262}$$

A calculation shows that formulas (134) and (211) hold for negative orders too, and we get [5, p. 255 10.9]

$$B^{(-n-p)}_{\mathrm{NWA},\nu,q}(x \oplus_q y) \ddot{=} (B^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q B^{(-p)}_{\mathrm{NWA},q}(y))^{\nu}, \tag{263}$$

and the same for first q-Euler polynomials.

A special case is the following q-analogue of [102, p. 139, (71)]

$$B^{(-n)}_{\mathrm{NWA},\nu,q}(x \oplus_q y) \ddot{=} (B^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q y)^{\nu}, \tag{264}$$

and the same for q-Euler polynomials.

Theorem 3.71. *A recurrence formula for the first q-Bernoulli numbers and a recurrence formula for the first q-Euler numbers.*

If $n, p \in \mathbb{Z}$ then

$$B^{(n+p)}_{\mathrm{NWA},\nu,q} \ddot{=} (B^{(n)}_{\mathrm{NWA},q} \oplus_q B^{(p)}_{\mathrm{NWA},q})^{\nu}, \tag{265}$$

$$F^{(n+p)}_{\mathrm{NWA},\nu,q} \ddot{=} (F^{(n)}_{\mathrm{NWA},q} \oplus_q F^{(p)}_{\mathrm{NWA},q})^{\nu}. \tag{266}$$

Theorem 3.72. *A q-analogue of* [102, p. 140 (72), (73)], [105, p. 1226, xvii]. *The first equation occurred in* [5, p. 255, 10.10].

$$(x \oplus_q y)^{\nu} \ddot{=} (B^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q B^{(n)}_{\mathrm{NWA},q}(y))^{\nu}, \tag{267}$$

$$(x \oplus_q y)^{\nu} \ddot{=} (E^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q E^{(n)}_{\mathrm{NWA},q}(y))^{\nu}. \tag{268}$$

Proof. Put $p = -n$ in (263). □

In particular for $y = 0$, we get a q-analogue of [105, p. 1226, xviii].

$$x^{\nu} \ddot{=} (B^{(-n)}_{\mathrm{NWA},q} \oplus_q B^{(n)}_{\mathrm{NWA},q}(x))^{\nu}, \tag{269}$$

$$x^{\nu} \ddot{=} (F^{(-n)}_{\mathrm{NWA},q} \oplus_q E^{(n)}_{\mathrm{NWA},q}(x))^{\nu}. \tag{270}$$

These recurrence formulas express first q-Bernoulli- and q-Euler polynomials of order n without mentioning polynomials of negative order.

These can also be expressed in the form

$$x^{\nu} = \sum_{s=0}^{\nu} \frac{B^{(-n)}_{\mathrm{NWA},s,q}}{\{s\}_q!} D_q^s B^{(n)}_{\mathrm{NWA},\nu,q}(x), \tag{271}$$

$$x^{\nu} = \sum_{s=0}^{\nu} \frac{F^{(-n)}_{\mathrm{NWA},s,q}}{\{s\}_q!} D_q^s E^{(n)}_{\mathrm{NWA},\nu,q}(x). \tag{272}$$

We conclude that the q-Bernoulli- and q-Euler polynomials satisfy linear q-difference equations with constant coefficients.

The following theorem is useful for the computation of q-Bernoulli- and q-Euler polynomials of positive order. This is because the polynomials of negative order are of simpler nature and can easily be computed. When the $B^{(-n)}_{\mathrm{NWA},s,q}$ etc. are known, (273) can be used to compute the $B^{(n)}_{\mathrm{NWA},s,q}$.

Theorem 3.73.

$$\sum_{s=0}^{\nu}\binom{\nu}{s}_q B^{(n)}_{\mathrm{NWA},s,q}B^{(-n)}_{\mathrm{NWA},\nu-s,q}=\delta_{\nu,0}. \tag{273}$$

$$\sum_{s=0}^{\nu}\binom{\nu}{s}_q F^{(n)}_{\mathrm{NWA},s,q}F^{(-n)}_{\mathrm{NWA},\nu-s,q}=\delta_{\nu,0}. \tag{274}$$

Proof. Put $x=y=0$ in (267) and (268). □

Theorem 3.74. *A q-analogue of* [102, p. 142]. *Assume that $f(x)$ is analytic with q-Taylor expansion*

$$f(x)=\sum_{\nu=0}^{\infty}D_q^{\nu}f(0)\frac{x^{\nu}}{\{\nu\}_q!}. \tag{275}$$

Then we can express powers of $\triangle_{\mathrm{NWA},q}$ and $\nabla_{\mathrm{NWA},q}$ operating on $f(x)$ as powers of D_q as follows. These series converge when the absolute value of x is small enough.

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\mathrm{NWA},q}} f(x)=\sum_{\nu=0}^{\infty}D_q^{\nu+n}f(0)\frac{B^{(-n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!}, \tag{276}$$

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\mathrm{NWA},q}} f(x)=\sum_{\nu=0}^{\infty}D_q^{\nu}f(0)\frac{E^{(-n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!}. \tag{277}$$

Proof. Use (134), (122) and (211), (122) respectively. □

Now put $f(x)=\mathrm{E}_q(xt)$ to obtain the generating function of the q-Bernoulli- and q-Euler polynomials of negative order.

$$\frac{\prod_{k=1}^n(\mathrm{E}_q(\omega_k t)-1)\mathrm{E}_q(xt)}{t^n\prod_{k=1}^n\omega_k}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}B^{(-n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n), \tag{278}$$

$$\frac{\prod_{k=1}^n(\mathrm{E}_q(\omega_k t)+1)\mathrm{E}_q(xt)}{2^n}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}E^{(-n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n). \tag{279}$$

The reason for the difference in appearance compared to the original for the following equation is that one of the function arguments is a Ward number.

Theorem 3.75. *A q-analogue of* [101, p. 191, (10)], [90, p. 21 (18)].

$$B^{(m)}_{\mathrm{NWA},\nu,q}(x \oplus_q \overline{n}_q) = \sum_{k=0}^{\min(\nu,n)} \binom{n}{k} \frac{\{\nu\}_q!}{\{\nu-k\}_q!} B^{(m-k)}_{\mathrm{NWA},\nu-k,q}(x). \tag{280}$$

Proof. Use (98) and (134). □

We can put $m = x = 0$ in (280) to obtain a q-analogue of [94, p. 133, (3)].

Theorem 3.76.

$$\frac{\{\nu\}_q!}{\{\nu-n\}_q!} x^{\nu-n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B^{(n)}_{\mathrm{NWA},\nu,q}(x \oplus_q \overline{k}_q). \tag{281}$$

Proof. Use equation (101). □

Theorem 3.77. *A q-analogue of* [101, (21) p. 163], [105, p. 1220]. *The corresponding formula for $n = 1$ occurred in* [5, p. 254].

$$\sum_{k=0}^{\infty} \frac{B^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \triangle^{n}_{\mathrm{NWA},q} D^k_q f(y) = D^n_{q,x} f(x \oplus_q y). \tag{282}$$

Proof. As in [101, p. 163] replace $f(x)$ by $f(x \oplus_q y)$ in (159).

$$f(B^{(n)}_{\mathrm{NWA},q}(x) \oplus_q y \oplus_q 1) - f(B^{(n)}_{\mathrm{NWA},q}(x) \oplus_q y) = D_q f(B^{(n-1)}_{\mathrm{NWA},q}(x) \oplus_q y). \tag{283}$$

Use the umbral formula (47) to get

$$\sum_{k=0}^{\infty} \frac{B^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \triangle_{\mathrm{NWA},q} D^k_q f(y) = \sum_{k=0}^{\infty} \frac{B^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} D^{k+1}_q f(y). \tag{284}$$

Apply the operator $\triangle^{n-1}_{\mathrm{NWA},q}$ with respect to y to both sides and use (276).

$$\sum_{k=0}^{\infty} \frac{B^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \triangle^{n}_{\mathrm{NWA},q} D^k_q f(y) = \sum_{k=0}^{\infty} \frac{B^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \sum_{l=0}^{\infty} D^{k+l+n}_q f(0) \frac{B^{(-n+1)}_{\mathrm{NWA},l,q}(y)}{\{l\}_q!}. \tag{285}$$

Finally use (47), (59), (267) to rewrite the righthand side. □

Remark 12. The RHS of (282) can also be written $D^n_{q,y} f(x \oplus_q y)$ or $D^n_q f(x \oplus_q y)$.

If we put $n = q = 1$ in (282), we get an Euler-Maclaurin expansion known from [94, p. 140]. If we also put $y = 0$, we get an expansion of a polynomial in terms of Bernoulli polynomials known from [81, p. 248].

Corollary 3.78. *A q-analogue of* [130]*. Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation*

$$\underset{\omega_1,\dots,\omega_n}{\triangle^n_{\mathrm{NWA},q}} f(x) = D^n_q \varphi(x) \tag{286}$$

is given by

$$f(x \oplus_q y) = \sum_{k=0}^{\nu} \frac{B^{(n)}_{\mathrm{NWA},k,q}(x|\omega_1,\dots,\omega_n)}{\{k\}_q!} D^k_q \varphi(y). \tag{287}$$

Proof. The LHS of (287) can be written as $\varphi(B^{(n)}_{\mathrm{NWA},q}(x|\omega_1,\dots,\omega_n) \oplus_q y)$, because if we apply $\triangle^n_{\mathrm{NWA},q,x}$ to both sides we get

$$\begin{aligned}&\underset{\omega_1,\dots,\omega_n}{\triangle^n_{\mathrm{NWA},q}} f(x \oplus_q y) = D^n_{q,x} \varphi(x \oplus_q y) = \\ &\underset{\omega_1,\dots,\omega_n}{\triangle^n_{\mathrm{NWA},q}} \varphi(B^{(n)}_{\mathrm{NWA},q}(x|\omega_1,\dots,\omega_n) \oplus_q y).\end{aligned} \tag{288}$$

□

Theorem 3.79. *A q-analogue of* [101, p. 156], [102, p. 127 (31)]*. Compare* [101, p. 147]*. A special case is found in* [81, p. 307].

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \nabla^n_{\mathrm{NWA},q} D^k_q f(y) = f(x \oplus_q y). \tag{289}$$

Proof. As in [101, p. 155] replace $f(x)$ by $f(x \oplus_q y)$ in (224).

$$\frac{1}{2}\Big(f(E^{(n)}_{\mathrm{NWA},q}(x) \oplus_q y \oplus_q 1) + f(E^{(n)}_{\mathrm{NWA},q}(x) \oplus_q y)\Big) = f(E^{(n-1)}_{\mathrm{NWA},q}(x) \oplus_q y). \tag{290}$$

Use the umbral formula (47) to get

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \nabla_{\mathrm{NWA},q} D^k_q f(y) = \sum_{k=0}^{\infty} \frac{E^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} D^k_q f(y). \tag{291}$$

Apply the operator $\nabla^{n-1}_{\mathrm{NWA},q}$ with respect to y to both sides and use (277).

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \nabla^n_{\mathrm{NWA},q} D^k_q f(y) = \sum_{k=0}^{\infty} \frac{E^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \sum_{l=0}^{\infty} D^{k+l}_q f(0) \frac{E^{(-n+1)}_{\mathrm{NWA},l,q}(y)}{\{l\}_q!}. \tag{292}$$

Finally use (47), (59), (268) to rewrite the righthand side. □

If we put $n = q = 1$ in (289), we get the Euler-Boole theorem known from [53, p. 128], [94, p. 149].

Corollary 3.80. *A q-analogue of* [130]*. Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation*

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}} f(x) = \varphi(x) \tag{293}$$

is given by

$$f(x \oplus_q y) = \sum_{k=0}^{\nu} \frac{E^{(n)}_{\text{NWA},k,q}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!} D^k_q \varphi(y). \tag{294}$$

Proof. The LHS of (294) can be written as $\varphi(E^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y)$, because if we apply $\nabla^n_{\text{NWA},q,x}$ to both sides we get

$$\begin{aligned}&\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}} f(x \oplus_q y) = \varphi(x \oplus_q y) = \\ &\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}} \varphi(E^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y).\end{aligned} \tag{295}$$

□

The following calculations are JHC q-analogues of Nörlund [102, p. 133 ff].

Definition 61. As a q-analogue of [102, (50) p. 133], [105, p. 1226, xvi] and [128, p. 378, (19)], we define second q-Bernoulli polynomials of two variables as

$$\begin{aligned}&B^{(n+p)}_{\text{JHC},\nu,q}(x \oplus_q y|\omega_1,\ldots,\omega_{n+p}) = \\ &\doteq (B^{(n)}_{\text{JHC},q}(x|\omega_1,\ldots,\omega_n) \oplus_q B^{(p)}_{\text{JHC},q}(y|\omega_{n+1},\ldots,\omega_{n+p}))^\nu,\end{aligned} \tag{296}$$

where we assume that n and p operate on x and y respectively, and the same for any $q-$polynomial.

The relation (182) together with (296) show, that $B^{(n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is a homogeneous function of $x,\omega_1,\ldots,\omega_n$ of degree ν, a q-analogue of [102, p. 134 (55)], i.e.

$$B^{(n)}_{\text{JHC},\nu,q}(\lambda x|\lambda\omega_1,\ldots,\lambda\omega_n) = \lambda^\nu B^{(n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n),\ \lambda \in \mathbb{C}. \tag{297}$$

And the same for second $q-$Euler-, Lucas- and $G-$ polynomials. This can be generalized in at least two ways in the same way as in the NWA case.

By (174) and (233) we get

$$\triangle^n_{\mathrm{JHC},q} B^{(n)}_{\mathrm{JHC},\nu,q}(x) = \frac{\{\nu\}_q!}{\{\nu - n\}_q!} x^{\nu-n},$$

$$\nabla^n_{\mathrm{JHC},q} E^{(n)}_{\mathrm{JHC},\nu,q}(x) = x^{\nu},$$

and we have

Definition 62. A q-analogue of [101, p. 177], [102, (66), p. 138]. The second q-Bernoulli polynomials of negative order $-n$ are given by

$$B^{(-n)}_{\mathrm{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n) \equiv \frac{\{\nu\}_q!}{\{\nu+n\}_q!} \underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\mathrm{JHC},q}} x^{\nu+n}, \tag{298}$$

and the second q-Euler polynomial of negative order $-n$ by the following q-analogue of [102, (67) p. 138]

$$E^{(-n)}_{\mathrm{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n) \equiv \underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\mathrm{JHC},q}} x^{\nu}, \tag{299}$$

where $\nu, n \in \mathbb{N}$. This defines second q-Bernoulli- and q-Euler polynomials of negative order as iterated $\triangle_{\mathrm{JHC},q}$ and $\nabla_{\mathrm{JHC},q}$ operating on positive integer powers of x.

Furthermore,

$$B^{(-n)}_{\mathrm{JHC},\nu,q} \equiv B^{(-n)}_{\mathrm{JHC},\nu,q}(0), \tag{300}$$

$$F^{(-n)}_{\mathrm{JHC},\nu,q} \equiv E^{(-n)}_{\mathrm{JHC},\nu,q}(0). \tag{301}$$

A calculation shows that formulas (174) and (233) hold for negative orders too, and we get

$$B^{(-n-p)}_{\mathrm{JHC},\nu,q}(x \oplus_q y) \doteq (B^{(-n)}_{\mathrm{JHC},q}(x) \oplus_q B^{(-p)}_{\mathrm{JHC},q}(y))^{\nu}, \tag{302}$$

and the same for q-Euler polynomials.

A special case is the following q-analogue of [102, p. 139, (71)]

$$B^{(-n)}_{\mathrm{JHC},\nu,q}(x \oplus_q y) \doteq (B^{(-n)}_{\mathrm{JHC},q}(x) \oplus_q y)^{\nu}, \tag{303}$$

and the same for the second q-Euler polynomials.

Theorem 3.81. *A recurrence formula for the second q-Bernoulli numbers and a recurrence formula for the second q-Euler numbers.*

If $n, p \in \mathbb{Z}$ then

$$B^{(n+p)}_{\mathrm{JHC},\nu,q} \doteq (B^{(n)}_{\mathrm{JHC},q} \oplus_q B^{(p)}_{\mathrm{JHC},q})^{\nu}, \tag{304}$$

$$F^{(n+p)}_{\mathrm{JHC},\nu,q} \doteq (F^{(n)}_{\mathrm{JHC},q} \oplus_q F^{(p)}_{\mathrm{JHC},q})^{\nu}. \tag{305}$$

Theorem 3.82. *A q-analogue of* [102, p. 140 (72), (73)], [105, p. 1226, xvii].

$$(x \oplus_q y)^\nu \ddot{=} (B^{(-n)}_{\mathrm{JHC},q}(x) \oplus_q B^{(n)}_{\mathrm{JHC},q}(y))^\nu, \tag{306}$$

$$(x \oplus_q y)^\nu \ddot{=} (E^{(-n)}_{\mathrm{JHC},q}(x) \oplus_q E^{(n)}_{\mathrm{JHC},q}(y))^\nu. \tag{307}$$

Proof. Put $p = -n$ in (302). □

In particular for $y = 0$, we get a q-analogue of [105, p. 1226, xviii].

$$x^\nu \ddot{=} (B^{(-n)}_{\mathrm{JHC},q} \oplus_q B^{(n)}_{\mathrm{JHC},q}(x))^\nu, \tag{308}$$

$$x^\nu \ddot{=} (F^{(-n)}_{\mathrm{JHC},q} \oplus_q E^{(n)}_{\mathrm{JHC},q}(x))^\nu. \tag{309}$$

These recurrence formulas express second q-Bernoulli- and q-Euler polynomials of order n without mentioning polynomials of negative order.

These can also be expressed in the form

$$x^\nu = \sum_{s=0}^{\nu} \frac{B^{(-n)}_{\mathrm{JHC},s,q}}{\{s\}_q!} D_q^s B^{(n)}_{\mathrm{JHC},\nu,q}(x), \tag{310}$$

$$x^\nu = \sum_{s=0}^{\nu} \frac{F^{(-n)}_{\mathrm{JHC},s,q}}{\{s\}_q!} D_q^s E^{(n)}_{\mathrm{JHC},\nu,q}(x). \tag{311}$$

We conclude that the second q-Bernoulli- and q-Euler polynomials satisfy linear q-difference equations with constant coefficients.

The following theorem is useful for the computation of second q-Bernoulli- and q-Euler polynomials of positive order. This is because the polynomials of negative order are of simpler nature and can easily be computed. When the $B^{(-n)}_{\mathrm{JHC},s,q}$ etc. are known, (312) can be used to compute the $B^{(n)}_{\mathrm{JHC},s,q}$.

Theorem 3.83.

$$\sum_{s=0}^{\nu} \binom{\nu}{s}_q B^{(n)}_{\mathrm{JHC},s,q} B^{(-n)}_{\mathrm{JHC},\nu-s,q} = \delta_{\nu,0}. \tag{312}$$

$$\sum_{s=0}^{\nu} \binom{\nu}{s}_q F^{(n)}_{\mathrm{JHC},s,q} F^{(-n)}_{\mathrm{JHC},\nu-s,q} = \delta_{\nu,0}. \tag{313}$$

Proof. Put $x = y = 0$ in (306) and (307). □

Theorem 3.84. *A q-analogue of* [102, p. 142]. *Assume that $f(x)$ is analytic with q-Taylor expansion*

$$f(x) = \sum_{\nu=0}^{\infty} D_q^\nu f(0) \frac{x^\nu}{\{\nu\}_q!}. \tag{314}$$

Then we can express powers of $\triangle_{\text{JHC},q}$ and $\nabla_{\text{JHC},q}$ operating on $f(x)$ as powers of D_q as follows. These series converge when the absolute value of x is small enough.

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\text{JHC},q}} f(x) = \sum_{\nu=0}^{\infty} D_q^{\nu+n} f(0) \frac{B^{(-n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!}, \tag{315}$$

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{JHC},q}} f(x) = \sum_{\nu=0}^{\infty} D_q^{\nu} f(0) \frac{E^{(-n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!}. \tag{316}$$

Proof. Use (174), (122) and (233), (122) respectively. □

Now put $f(x) = \mathrm{E}_q(xt)$ to obtain the generating function of the second q-Bernoulli- and q-Euler polynomials of negative order.

$$\frac{\prod_{k=1}^n (\mathrm{E}_{\frac{1}{q}}(\omega_k t) - 1)\mathrm{E}_q(xt)}{t^n \prod_{k=1}^n \omega_k} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} B^{(-n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n), \tag{317}$$

$$\frac{\prod_{k=1}^n (\mathrm{E}_{\frac{1}{q}}(\omega_k t) + 1)\mathrm{E}_q(xt)}{2^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} E^{(-n)}_{\text{JHC},\nu,q}(x|\omega_1,\ldots,\omega_n). \tag{318}$$

The reason for the difference in appearance compared to the original for the following equation is that one of the function arguments is a Jackson number.

Theorem 3.85. *A q-analogue of* [101, p. 191, (10)]*,* [90, p. 21 (18)].

$$B^{(m)}_{\text{JHC},\nu,q}(x \boxplus_q \tilde{n}_q) = \sum_{k=0}^{\min(\nu,n)} \binom{n}{k} \frac{\{\nu\}_q!}{\{\nu-k\}_q!} B^{(m-k)}_{\text{JHC},\nu-k,q}(x). \tag{319}$$

Proof. Use (109) and (174). □

We can put $m = x = 0$ in (319) to obtain a q-analogue of [94, p. 133, (3)].

Theorem 3.86.

$$\frac{\{\nu\}_q!}{\{\nu-n\}_q!} x^{\nu-n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B^{(n)}_{\text{JHC},\nu,q}(x \boxplus_q \tilde{k}_q). \tag{320}$$

Proof. Use equation (112). □

Theorem 3.87. *A q-analogue of* [101, (21) p. 163]*,* [105, p. 1220].

$$\sum_{k=0}^{\infty} \frac{B^{(n)}_{\text{JHC},k,q}(x)}{\{k\}_q!} \triangle^n_{\text{JHC},q} D_q^k f(y) = D^n_{q,x} f(x \oplus_q y). \tag{321}$$

Proof. As in [101, p. 163] replace $f(x)$ by $f(x \oplus_q y)$ in (201).

$$f(B^{(n)}_{\text{JHC},q}(x) \oplus_q y \boxplus_q 1) - f(B^{(n)}_{\text{JHC},q}(x) \oplus_q y) = D_q f(B^{(n-1)}_{\text{JHC},q}(x) \oplus_q y). \quad (322)$$

Use the umbral formula (47) to get

$$\sum_{k=0}^{\infty} \frac{B^{(n)}_{\text{JHC},k,q}(x)}{\{k\}_q!} \triangle_{\text{JHC},q} D_q^k f(y) = \sum_{k=0}^{\infty} \frac{B^{(n-1)}_{\text{JHC},k,q}(x)}{\{k\}_q!} D_q^{k+1} f(y). \quad (323)$$

Apply the operator $\triangle^{n-1}_{\text{JHC},q}$ with respect to y to both sides and use (315).

$$\sum_{k=0}^{\infty} \frac{B^{(n)}_{\text{JHC},k,q}(x)}{\{k\}_q!} \triangle^n_{\text{JHC},q} D_q^k f(y) = \sum_{k=0}^{\infty} \frac{B^{(n-1)}_{\text{JHC},k,q}(x)}{\{k\}_q!} \sum_{l=0}^{\infty} D_q^{k+l+n} f(0) \frac{B^{(-n+1)}_{\text{JHC},l,q}(y)}{\{l\}_q!}. \quad (324)$$

Finally use (47), (59), (306) to rewrite the righthand side. □

If we put $n = q = 1$ in (321), we get an Euler-Maclaurin expansion known from [94, p. 140]. If we also put $y = 0$, we get an expansion of a polynomial in terms of Bernoulli polynomials known from [81, p. 248].

Corollary 3.88. *A q-analogue of* [130]. *Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation*

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\text{JHC},q}} f(x) = D_q^n \varphi(x) \quad (325)$$

is given by

$$f(x \oplus_q y) = \sum_{k=0}^{\nu} \frac{B^{(n)}_{\text{JHC},k,q}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!} D_q^k \varphi(y). \quad (326)$$

Theorem 3.89. *A q-analogue of* [101, p. 156], [102, p. 127 (31)]. *Compare* [101, p. 147]. *A special case is found in* [81, p. 307].

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\text{JHC},k,q}(x)}{\{k\}_q!} \nabla^n_{\text{JHC},q} D_q^k f(y) = f(x \oplus_q y). \quad (327)$$

Proof. As in [101, p. 155] replace $f(x)$ by $f(x \oplus_q y)$ in (246).

$$\frac{1}{2}\Big(f(E^{(n)}_{\text{JHC},q}(x) \oplus_q y \boxplus_q 1) + f(E^{(n)}_{\text{JHC},q}(x) \oplus_q y)\Big) = f(E^{(n-1)}_{\text{JHC},q}(x) \oplus_q y). \quad (328)$$

Use the umbral formula (47) to get

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\text{JHC},k,q}(x)}{\{k\}_q!} \nabla_{\text{JHC},q} D_q^k f(y) = \sum_{k=0}^{\infty} \frac{E^{(n-1)}_{\text{JHC},k,q}(x)}{\{k\}_q!} D_q^k f(y). \quad (329)$$

Apply the operator $\nabla_{\mathrm{JHC},q}^{n-1}$ with respect to y to both sides and use (316).

$$\sum_{k=0}^{\infty}\frac{E_{\mathrm{JHC},k,q}^{(n)}(x)}{\{k\}_q!}\nabla_{\mathrm{JHC},q}^{n}D_q^k f(y)=\sum_{k=0}^{\infty}\frac{E_{\mathrm{JHC},k,q}^{(n-1)}(x)}{\{k\}_q!}\sum_{l=0}^{\infty}D_q^{k+l}f(0)\frac{E_{\mathrm{JHC},l,q}^{(-n+1)}(y)}{\{l\}_q!}. \tag{330}$$

Finally use (47), (59), (307) to rewrite the righthand side. □

If we put $n = q = 1$ in (327), we get the Euler-Boole theorem known from [53, p. 128], [94, p. 149].

Corollary 3.90. *A q-analogue of* [130]. *Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation*

$$\underset{\omega_1,\ldots,\omega_n}{\nabla_{\mathrm{JHC},q}^{n}}\; f(x)=\varphi(x) \tag{331}$$

is given by

$$f(x\oplus_q y)=\sum_{k=0}^{\nu}\frac{E_{\mathrm{JHC},k,q}^{(n)}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!}D_q^k\varphi(y). \tag{332}$$

There are a few formulas similar to the Leibniz theorem. We can express the NWA difference operator in terms of the mean value operator and vice versa.

Theorem 3.91.

$$\triangle_{\mathrm{NWA},q}^{n}(fg)=2^n\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\nabla_{\mathrm{NWA},q}^{i}f(\nabla_{\mathrm{NWA},q}^{n-i}E(\oplus_q)^i)g. \tag{333}$$

Proof. Same as [81, p. 98, (13)]. □

Theorem 3.92.

$$\nabla_{\mathrm{NWA},q}^{n}(fg)=(-\frac{1}{2})^n\sum_{i=0}^{n}(-2)^{i}\binom{n}{i}\nabla_{\mathrm{NWA},q}^{i}f(\triangle_{\mathrm{NWA},q}^{n-i}E(\oplus_q)^i)g. \tag{334}$$

Proof. Same as [81, p. 99, (2)]. □

Theorem 3.93.

$$\nabla_{\mathrm{NWA},q}^{n}(fg)=\sum_{i=0}^{n}(\frac{1}{2})^{i}\binom{n}{i}\triangle_{\mathrm{NWA},q}^{i}f(\nabla_{\mathrm{NWA},q}^{n-i}E(\oplus_q)^i)g. \tag{335}$$

Proof. Same as [81, p. 99, (3)]. □

Theorem 3.94. *A q-analogue of Lagrange 1772,* [81, p. 101], [28, p. 19], [94, p. 37]. *The inverse NWA difference is given by*

$$\triangle_{\mathrm{NWA},q}^{-1} f(\overline{x_q})|_0^n = \sum_{k=0}^{n-1} f(\overline{k_q}) \equiv \sum_0^n f(\overline{x_q})\delta_q(x). \tag{336}$$

Proof. Use the same idea as Euler, reproduced by F. Schweins [121, p. 9]. □

Theorem 3.95. *The inverse ∇ is given by*

$$\nabla_{\mathrm{NWA},q}^{-1}\left(\frac{f(\overline{0_q}) + (-1)^{n-1} f(\overline{n_q})}{2}\right) = \sum_{k=0}^{n-1} (-1)^k f(\overline{k_q}). \tag{337}$$

Theorem 3.96. *The analogue of integration by parts is a q-analogue of* [81, p. 105], [28, p. 21], [94, p. 41], [103, p. 19].

$$\sum_{k=0}^{n-1} f(\overline{k_q})\triangle_{\mathrm{NWA},q} g(\overline{k_q}) = [f(\overline{x_q})g(\overline{x_q})]_0^n - \sum_{k=0}^{n-1} E(\oplus_q)g(\overline{k_q})\triangle_{\mathrm{NWA},q} f(\overline{k_q}). \tag{338}$$

Theorem 3.97. *A q-analogue of Euler's symbolic formula* [51, p. 303], [89, p. 242, (1)].

$$\sum_{k=0}^{n-1} f(\overline{k_q}) \doteq \int_{\overline{0}_q}^{\overline{n}_q} f(x \oplus_q B_{\mathrm{NWA},q})\, d_q(x) \equiv \int_{\overline{0}_q}^{\overline{n}_q} f(B_{\mathrm{NWA},q}(x))\, d_q(x) \equiv \int_{B_{\mathrm{NWA},q}}^{B_{\mathrm{NWA},q} \oplus_q \overline{n}_q} f(x)\, d_q(x). \tag{339}$$

Proof. Apply $\triangle_{\mathrm{NWA},q}$ to both sides to get

$$f(x)|_{\overline{0_q}}^{\overline{n_q}} \doteq \triangle_{\mathrm{NWA},q} \int_{\overline{0}_q}^{\overline{n}_q} f(x \oplus_q B_{\mathrm{NWA},q})\, d_q(x). \tag{340}$$

Then apply (138). □

Corollary 3.98. *A q-analogue of* [113, p. 701].

$$J_{\mathrm{B,N},q} f(x \oplus_q B_{\mathrm{NWA},q}) \doteq f(x). \tag{341}$$

We immediately get a proof of the following formula, which is of considerable use in integration theory.

Theorem 3.99. *The q-Euler-Maclaurin summation theorem for formal power series. A q-analogue of* [28, p. 54], [51, p. 303], [103, p. 25], [113, p. 706], [81, p. 253].

$$\sum_{k=0}^{n-1} f(\overline{k_q}) = \int_{\overline{0}_q}^{\overline{n}_q} f(x)\, d_q(x) - \frac{1}{\{2\}_q}(f(\overline{n}_q) - f(0)) + \sum_{k=2}^{\infty} \frac{B_{\text{NWA},k,q}}{\{k\}_q!}(D_q^{k-1} f(\overline{n}_q) - D_q^{k-1} f(0)). \tag{342}$$

Example 4. Put $f(x) = x^m$ in (342) to get formula (162).

Corollary 3.100. *A dual to the q-Euler-Maclaurin summation theorem (342). The q-integral on the RHS shall be interpreted in the following way: First q-integrate f in the form $f(x)$. Then put in the values in the umbral sense according to (47).*

$$\sum_{k=0}^{n-1} f(\overline{k_q}) \ddot{=} \int_{0}^{\overline{n}_q} f(B_{\text{NWA},q})\, d_q(x) + \sum_{k=0}^{\infty} \frac{(\overline{x_q})^{k+1}}{\{k+1\}_q!} D_q^k f(B_{\text{NWA},k,q})|_0^n. \tag{343}$$

We will now derive analogous results for q-Euler numbers. We start with

Theorem 3.101. *A generalization of (230). Compare* [53, p. 136] *($q = 1$).*

$$\sum_{k=0}^{n-1} (-1)^k f(\overline{k_q}) \ddot{=} \frac{(-1)^{x-1}}{2} f(E_{\text{NWA},q}(x))|_{\overline{0_q}}^{\overline{n_q}}. \tag{344}$$

Proof. Apply $\nabla_{\text{NWA},q}$ to both sides to get by (337)

$$\frac{f(\overline{0_q}) + (-1)^{n-1} f(\overline{n_q})}{2} \ddot{=} \nabla_{\text{NWA},q} \frac{(-1)^{x-1}}{2} f(E_{\text{NWA},q}(x))|_{\overline{0_q}}^{\overline{n_q}}. \tag{345}$$

Finally use (224) with $n = 1$. □

Theorem 3.102. *Almost a q-analogue of Boole's first summation formula for alternate functions* [81, p. 316, (2)]. *A q-analogue of the Euler formula* [51, p. 310], [89, p. 252].

$$\sum_{k=0}^{n-1} (-1)^k f(\overline{k_q}) = \sum_{k=0}^{\infty} \frac{F_{\text{NWA},k,q}}{\{k\}_q!} D_q^k \left(\frac{f(\overline{0}_q)}{2} + (-1)^{n-1} \frac{f(\overline{n}_q)}{2} \right) \ddot{=} \frac{1}{2} \left[f(F_{\text{NWA},q}) + (-1)^{n-1} f(F_{\text{NWA},q} \oplus_q \overline{n_q}) \right]. \tag{346}$$

Proof. Use (289). □

There are some further formulas similar to the Leibniz theorem. We can express the JHC difference operator in terms of the mean value operator and vice versa.

Theorem 3.103.

$$\triangle_{\text{JHC},q}^{n}(fg)=2^{n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\nabla_{\text{JHC},q}^{i}f(\nabla_{\text{JHC},q}^{n-i}E(\boxplus_{q})^{i})g. \tag{347}$$

Proof. Same as [81, p. 98, (13)]. □

Theorem 3.104.

$$\nabla_{\text{JHC},q}^{n}(fg)=(-\frac{1}{2})^{n}\sum_{i=0}^{n}(-2)^{i}\binom{n}{i}\nabla_{\text{JHC},q}^{i}f(\triangle_{\text{JHC},q}^{n-i}E(\boxplus_{q})^{i})g. \tag{348}$$

Proof. Same as [81, p. 99, (2)]. □

Theorem 3.105.

$$\nabla_{\text{JHC},q}^{n}(fg)=\sum_{i=0}^{n}(\frac{1}{2})^{i}\binom{n}{i}\triangle_{\text{JHC},q}^{i}f(\nabla_{\text{JHC},q}^{n-i}E(\boxplus_{q})^{i})g. \tag{349}$$

Proof. Same as [81, p. 99, (3)]. □

Theorem 3.106. *A q-analogue of* [81, p. 101], [28, p. 19], [94, p. 37]. *The inverse JHC difference is given by*

$$\triangle_{\text{JHC},q}^{-1}f(\tilde{x_{q}})|_{0}^{n}=\sum_{k=0}^{n-1}f(\tilde{k_{q}})\equiv\sum_{0}^{n}f(\tilde{x_{q}})\delta_{q}(x). \tag{350}$$

Proof. Use the same idea as Euler, reproduced by F. Schweins [121, p. 9]. □

Theorem 3.107. *The inverse ∇ is given by*

$$\nabla_{\text{JHC},q}^{-1}\left(\frac{f(\tilde{0_{q}})+(-1)^{n-1}f(\tilde{n_{q}})}{2}\right)=\sum_{k=0}^{n-1}(-1)^{k}f(\tilde{k_{q}}). \tag{351}$$

Theorem 3.108. *The analogue of integration by parts is a q-analogue of* [81, p. 105], [28, p. 21], [94, p. 41], [103, p. 19].

$$\sum_{k=0}^{n-1}f(\tilde{k_{q}})\triangle_{\text{JHC},q}g(\tilde{k_{q}})=[f(\tilde{x_{q}})g(\tilde{x_{q}})]_{0}^{n}-\sum_{k=0}^{n-1}E(\boxplus_{q})g(\tilde{k_{q}})\triangle_{\text{JHC},q}f(\tilde{k_{q}}). \tag{352}$$

Theorem 3.109. *A q-analogue of Eulers symbolic formula* [51, p. 303], [89, p. 242, (1)].

$$\begin{aligned}&\sum_{k=0}^{n-1} f(\tilde{k_q}) \doteq \int_{\tilde{0}_q}^{\tilde{n}_q} f(x \oplus_q B_{\mathrm{JHC},q})\, d_q(x) \equiv \int_{\tilde{0}_q}^{\tilde{n}_q} f(B_{\mathrm{JHC},q}(x))\, d_q(x)\\ &\equiv \int_{B_{\mathrm{JHC},q}}^{B_{\mathrm{JHC},q}\oplus_q \tilde{n}_q} f(x)\, d_q(x).\end{aligned} \tag{353}$$

Proof. Apply $\triangle_{\mathrm{JHC},q}$ to both sides to get

$$f(x)|_{\tilde{0}_q}^{\tilde{n}_q} = \triangle_{\mathrm{JHC},q} \int_{\tilde{0}_q}^{\tilde{n}_q} f(x \oplus_q B_{\mathrm{JHC},q})\, d_q(x). \tag{354}$$

Then apply (178). ∎

Corollary 3.110. *A q-analogue of* [113, p. 701].

$$J_{\mathrm{B,J},q} f(x \oplus_q B_{\mathrm{JHC},q}) \doteq f(x). \tag{355}$$

We immediately get a proof of the following formula, which is of considerable use in integration theory.

Theorem 3.111. *The q-Euler-Maclaurin summation theorem for formal power series. A q-analogue of* [28, p. 54], [51, p. 303], [103, p. 25], [113, p. 706], [81, p. 253].

$$\begin{aligned}&\sum_{k=0}^{n-1} f(\tilde{k_q}) = \int_{\tilde{0}_q}^{\tilde{n}_q} f(x)\, d_q(x) -\\ &\frac{1}{\{2\}_q}(f(\tilde{n}_q) - f(0)) + \sum_{k=2}^{\infty} \frac{B_{\mathrm{JHC},k,q}}{\{k\}_q!}(D_q^{k-1} f(\tilde{n}_q) - D_q^{k-1} f(0)).\end{aligned} \tag{356}$$

Example 5. Put $f(x) = x^m$ in (356) to get formula (204).

Corollary 3.112. *A dual to the q-Euler-Maclaurin summation theorem (356). The q-integral on the RHS shall be interpreted in the following way: First q-integrate f in the form f(x). Then put in the values in the umbral sense according to (47).*

$$\begin{aligned}&\sum_{k=0}^{n-1} f(\tilde{k_q}) \doteq \int_{0}^{\tilde{n}_q} f(B_{\mathrm{JHC},q})\, d_q(x) +\\ &\sum_{k=0}^{\infty} \frac{(\tilde{x_q})^{k+1}}{\{k+1\}_q!} D_q^k f(B_{\mathrm{JHC},k,q})|_0^n.\end{aligned} \tag{357}$$

We will now derive an analogous result for the second q-Euler numbers. We start with

Theorem 3.113. *A generalization of (252). Compare* [53, p. 136] *($q = 1$).*

$$\sum_{k=0}^{n-1}(-1)^k f(\tilde{k_q}) \ddot{=} \frac{(-1)^{x-1}}{2} f(E_{\mathrm{JHC},q}(x))\Big|_{\tilde{0_q}}^{\tilde{n_q}}. \tag{358}$$

Proof. Apply $\nabla_{\mathrm{JHC},q}$ to both sides to get by (351)

$$\frac{f(\tilde{0_q}) + (-1)^{n-1} f(\tilde{n_q})}{2} = \nabla_{\mathrm{JHC},q} \frac{(-1)^{x-1}}{2} f(E_{\mathrm{JHC},q}(x))\Big|_{\tilde{0_q}}^{\tilde{n_q}}. \tag{359}$$

Finally use (246) with $n = 1$. □

We will continue this part with a theorem involving both q-Bernoulli and q-Euler polynomials.

Theorem 3.114. *A q-analogue of* [128, p. 379].

$$B_{\mathrm{NWA},\nu,q}^{(n)}(x \oplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \left(B_{\mathrm{NWA},k,q}^{(n)}(y) + \frac{\{k\}_q}{2} B_{\mathrm{NWA},k-1,q}^{(n-1)}(y) \right) E_{\mathrm{NWA},\nu-k,q}(x). \tag{360}$$

Theorem 3.115. *A q-analogue of* [81, p. 241]*. The mean of a q-Bernoulli polynomial with Ward number argument is given by*

$$\nabla_{\mathrm{NWA},q} B_{\mathrm{NWA},\nu,q}(\overline{x_q}) = B_{\mathrm{NWA},\nu,q}(\overline{x_q}) + \frac{\{\nu\}_q}{2} (\overline{x_q})^{\nu-1}. \tag{361}$$

Theorem 3.116. *This implies a formula for the inverse of the mean, a q-analogue of* [81, p. 241, (3)].

$$\nabla_{\mathrm{NWA},q}^{-1} B_{\mathrm{NWA},\nu,q}(\overline{x_q}) = B_{\mathrm{NWA},\nu,q}(\overline{x_q}) - \frac{\{\nu\}_q}{2} E_{\mathrm{NWA},\nu-1,q}(\overline{x_q}). \tag{362}$$

The proofs of the following formulas are made through the generating function. Observe that we have to change to JHC on the LHS.

Theorem 3.117. *A q-analogue of the Bernoulli complementary argument theorem* [107, p. 354]*,* [94, p. 128, (1)].

$$B_{\mathrm{JHC},\nu,q}(x) = (-1)^{\nu} B_{\mathrm{NWA},\nu,q}(1 \ominus_q x). \tag{363}$$

Theorem 3.118. *A q-analogue of the Euler complementary argument theorem* [94, p. 145, (1)].

$$E_{\mathrm{JHC},\nu,q}(x) = (-1)^{\nu} E_{\mathrm{NWA},\nu,q}(1 \ominus_q x). \tag{364}$$

4. q-Lucas and $q-G$ polynomials

The process of computations based on q-Bernoulli polynomials and q-Euler polynomials can be continued in the way that is shown in this chapter. Further generalizations are possible, but we have not pursued this path.

The following definition is reminiscent of [102, p. 6, (12)].

Definition 63.

$$\triangle_{\text{NWA},2,q} \equiv 2\triangle_{\text{NWA},q}\nabla_{\text{NWA},q} \equiv E(\oplus_q)^{\overline{2}_q} - I. \tag{365}$$

$$\nabla_{\text{NWA},2,q} \equiv \frac{E(\oplus_q)^{\overline{2}_q} + I}{2}. \tag{366}$$

This can be generalized to

Definition 64.

$$\underset{\omega}{\triangle_{\text{NWA},2,q}} \equiv \frac{E(\oplus_q)^{\overline{2}_q\omega} - I}{2\omega}, \ \omega \in \mathbb{C}. \tag{367}$$

$$\underset{\omega}{\nabla_{\text{NWA},2,q}} \equiv \frac{E(\oplus_q)^{\overline{2}_q\omega} + I}{2}, \ \omega \in \mathbb{C}. \tag{368}$$

The following formulas apply just as in the previous case.

Theorem 4.1.

$$\nabla^n_{\text{NWA},q}\triangle^n_{\text{NWA},q} f(x) = 2^{-n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} f(x \oplus_q \overline{2k}_q). \tag{369}$$

$$f(\overline{2n}_q) = \sum_{k=0}^{n}\binom{n}{k}\triangle^k_{\text{NWA},2,q} f(0). \tag{370}$$

$$\nabla^n_{\text{NWA},2,q} f(x) = 2^{-n}\sum_{k=0}^{n}\binom{n}{k} f(x \oplus_q \overline{2k}_q). \tag{371}$$

Definition 65. The generating function for the q-Lucas polynomial $L^{(n)}_{\text{NWA},\nu,q}(x)$ is

$$\frac{(2t)^n}{(\mathrm{E}_q(t\overline{2}_q) - 1)^n}\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty}\frac{t^\nu L^{(n)}_{\text{NWA},\nu,q}(x)}{\{\nu\}_q!}, \ |t| < \pi. \tag{372}$$

This can be generalized to

Definition 66. The generating function for $L_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)$ is

$$\frac{(2t)^n\omega_1\ldots\omega_n}{\prod_{k=1}^n(\mathrm{E}_q(\omega_k t\overline{2}_q)-1)}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu L_{NWA,\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!},\quad |t|<\min(|\frac{\pi}{\omega_1}|,\ldots,|\frac{\pi}{\omega_n}|). \tag{373}$$

Obviously, $L_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$L_{\mathrm{NWA},\nu,q}^{(1)}(x|\omega)=\omega^\nu L_{\mathrm{NWA},\nu,q}(\frac{x}{\omega}). \tag{374}$$

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\mathrm{NWA},2,q}}\; L_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)=\{\nu\}_q x^{\nu-1}. \tag{375}$$

Theorem 4.2. *The successive differences of q-Lucas polynomials can be expressed as q-Lucas polynomials.*

$$\underset{\omega_1,\ldots,\omega_p}{\triangle^p_{\mathrm{NWA},2,q}}\; L_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)=\frac{\{\nu\}_q!}{\{\nu-p\}_q!}L_{\mathrm{NWA},\nu-p,q}^{(n-p)}(x|\omega_{p+1},\ldots,\omega_n), \tag{376}$$

$$L_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)\ddot{=}(L_{\mathrm{NWA},\nu,\omega_1,\ldots,\omega_n,q}^{(n)}\oplus_q x)^\nu. \tag{377}$$

The following invertible operator will be useful in this connection.

Definition 67. Compare [28, p. 32] ($n=1$). The operator $S_{\mathrm{L,N},q}^n\in\mathbb{C}(D_q)$ is given by

$$S_{\mathrm{L,N},q}^n\equiv\frac{(\mathrm{E}_q(\overline{2_q}D_q)-I)^n}{(2D_q)^n}. \tag{378}$$

This implies

Theorem 4.3.

$$\nabla_{\mathrm{NWA},q}^n\triangle_{\mathrm{NWA},q}^n=D_q^n S_{\mathrm{L,N},q}^n. \tag{379}$$

Theorem 4.4. *Compare* [28, p. 43, 3.3]

$$\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}L_{\mathrm{NWA},\nu,q}^{(n)}(x\oplus_q\overline{2k}_q)=\{\nu-n+1\}_{n,q}x^{\nu-n}2^n. \tag{380}$$

Proof.

$$\begin{aligned}&\triangle_{\mathrm{NWA},2,q}^n L_{\mathrm{NWA},\nu,q}^{(n)}(x)=D_q^n S_{\mathrm{L,N},q}^n L_{\mathrm{NWA},\nu,q}^{(n)}(x)2^n==\\&D_q^n x^\nu 2^n=\{\nu-n+1\}_{n,q}x^{\nu-n}2^n.\end{aligned} \tag{381}$$

□

Corollary 4.5.

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}L_{\text{NWA},\nu,q}^{(n)}(\overline{2k}_q)=\{n\}_q!\delta_{0,\nu-n}2^n. \tag{382}$$

Proof. Put $x=0$ in (380). □

By the generating function

$$\frac{1}{2}\triangle_{\text{NWA},2,q}L_{\text{NWA},\nu,q}^{(n)}(x)=\{\nu\}_q L_{\text{NWA},\nu-1,q}^{(n-1)}(x)=D_q L_{\text{NWA},\nu,q}^{(n-1)}(x). \tag{383}$$

The following symbolic relations obtain.

Theorem 4.6.

$$\frac{1}{2}\left[(L_{\text{NWA},q}^{(n)}\oplus_q x\oplus_q\overline{2}_q)^{\nu}-(L_{\text{NWA},q}^{(n)}\oplus_q x)^{\nu}\right]\doteqdot\{\nu\}_q(L_{\text{NWA},q}^{(n-1)}\oplus_q x)^{\nu-1}. \tag{384}$$

$$\frac{1}{2}\left[(L_{\text{NWA},q}^{(n)}\oplus_q\overline{2}_q)^{\nu}-L_{\text{NWA},\nu,q}^{(n)}\right]\doteqdot\{\nu\}_q L_{\text{NWA},\nu-1,q}^{(n-1)}. \tag{385}$$

Theorem 4.7.

$$\begin{aligned}&\triangle_{\text{NWA},q}\nabla_{\text{NWA},q}f(L_{\text{NWA},\nu,q}^{(n)}(x))\equiv\\&\frac{1}{2}\left[f(L_{\text{NWA},\nu,q}^{(n)}(x)\oplus_q\overline{2}_q)-f(L_{\text{NWA},\nu,q}^{(n)}(x))\right]\doteqdot D_q f(L_{\text{NWA},\nu,q}^{(n-1)}(x)).\end{aligned} \tag{386}$$

Theorem 4.8. *The following recurrence obtains.*

$$\frac{1}{2}\left[(L_{\text{NWA},q}\oplus_q\overline{2}_q)^k-L_{\text{NWA},k,q}\right]\doteqdot\delta_{1,k}. \tag{387}$$

Theorem 4.9.

$$\frac{1}{2}\sum_{k=1}^{\nu}\binom{\nu}{k}_q L_{\text{NWA},\nu-k,q}^{(n)}(x)(\overline{2}_q)^k=\{\nu\}_q L_{\text{NWA},\nu-1,q}^{(n-1)}(x). \tag{388}$$

Proof. Use equation (384). □

Theorem 4.10. *The q-Lucas polynomials of degree ν and order n can be expressed as*

$$L_{\text{NWA},\nu,q}^{(n)}(t)=S_{\text{L,N},q}^{-n}t^{\nu}. \tag{389}$$

Proof.

$$LHS=\sum_{k=0}^{\nu}\binom{\nu}{k}_q L_{\text{NWA},k,q}^{(n)}t^{\nu-k}=\sum_{k=0}^{\infty}\frac{L_{\text{NWA},k,q}^{(n)}}{\{k\}_q!}D_q^k t^{\nu}\overset{\text{by}(372)}{=}RHS. \tag{390}$$

□

Theorem 4.11. *Compare* [28, 3.15 p. 51]*, where the corresponding formula for Euler polynomials was given.*

$$L_{\mathrm{NWA},\nu,q}(x) \equiv \frac{\{\nu\}_q}{\mathrm{E}_q(\overline{2}_q D_q) - I} x^{\nu-1} = \frac{\{\nu\}_q}{E(\oplus_q)^{\overline{2}_q} - I} x^{\nu-1} \doteq (x \oplus_q L_{\mathrm{NWA},q})^{\nu}. \tag{391}$$

The first q-Lucas numbers have the following values.

$$L_{\mathrm{NWA},0,q} = 1; L_{\mathrm{NWA},1,q} = (-3-q)(2+2q)^{-1} \tag{392}$$

$$L_{\mathrm{NWA},2,q} = (1+3q+8q^2+3q^3+q^4)(4+8q+8q^2+4q^3)^{-1} \tag{393}$$

$$L_{\mathrm{NWA},3,q} = (-1-2q-2q^2-9q^3+9q^4+2q^5+2q^6+q^7)[8(q+1)^2(q^2+1)]^{-1}. \tag{394}$$

$$L_{\mathrm{NWA},4,q} = f(q)(16(1+q)^2\{3\}_q\{5\}_q)^{-1}, \tag{395}$$

where

$$\begin{aligned} f(q) = 1 &+ 3q + q^2 - 11q^3 - 18q^4 - 63q^5 - 104q^6 - 130q^7 \\ &- 104q^8 - 63q^9 - 18q^{10} - 11q^{11} + q^{12} + 3q^{13} + q^{14}. \end{aligned} \tag{396}$$

Theorem 4.12. *We have the following operational representation.*

$$L^{(n)}_{\mathrm{NWA},\nu,q}(\omega_1, \ldots, \omega_n) \doteq (\oplus_{q,l=1}^{n} \omega_l L_{\mathrm{NWA},l,q})^{\nu}. \tag{397}$$

Corollary 4.13.

$$\mathrm{E}_q(tL_{\mathrm{NWA},q}) \doteq \frac{2t}{\mathrm{E}_q(t\overline{2}_q) - 1}. \tag{398}$$

We will now give a few other equations for q-Lucas polynomials. We start with

Definition 68. A q-analogue of [138, p. 575], [98, p. 401 (7)], [89, p. 253].

$$t_{\mathrm{NWA},m,q}(n) \equiv \sum_{k=0}^{n-1} (\overline{2k+1_q})^m. \tag{399}$$

Theorem 4.14. *A q-analogue of* [89, p. 261].

$$t_{\mathrm{NWA},m,q}(n) = \frac{L_{\mathrm{NWA},m+1,q}(\overline{(2n+1)}_q) - L_{\mathrm{NWA},m+1,q}(\overline{(1)}_q)}{\{m+1\}_q}. \tag{400}$$

Proof.

$$\begin{aligned} LHS = \sum_{k=0}^{n-1} L^{(0)}_{\mathrm{NWA},m,q}(\overline{2k+1_q}) = \\ \sum_{k=0}^{n-1} \frac{1}{\{m+1\}_q} \triangle_{\mathrm{NWA},2,q} L_{\mathrm{NWA},m+1,q}(\overline{2k+1_q}) = RHS. \end{aligned} \tag{401}$$

□

Theorem 4.15.

$$2x^n = \int_x^{x\oplus_q \overline{2_q}} L_{\text{NWA},n,q}(t)\, d_q(t) = \frac{L_{\text{NWA},n+1,q}(x \oplus_q \overline{2_q}) - L_{\text{NWA},n+1,q}(x)}{\{n+1\}_q}. \tag{402}$$

Proof. q-integrate (123) for $n = 1$ and use (383). □

This can be rewritten as

$$2x^n = \frac{1}{\{n+1\}_q} \sum_{k=0}^{n} \binom{n+1}{k}_q L_{\text{NWA},k,q}(x)(\overline{2_q})^{n+1-k}. \tag{403}$$

Theorem 4.16.

$$J_{\text{L,N},q} \equiv \frac{\triangle_{\text{NWA},2,q}}{2D_q} f(x) = \int_x^{x\oplus_q \overline{2}_q} f(t)\, d_q(t). \tag{404}$$

Proof. By (389) and (402) we have

$$\int_x^{x\oplus_q \overline{2_q}} L_{\text{NWA},n,q}(t)\, d_q(t) = \frac{\triangle_{\text{NWA},2,q}}{2D_q} L_{\text{NWA},n,q}(x). \tag{405}$$

Now any formal power series can be expanded as a sum of $L_{\text{NWA},n,q}(x)$. □

Theorem 4.17.

$$f(x) = \sum_{k=0}^{\infty} \int_{\overline{0}_q}^{\overline{2}_q} D_q^k f(t)\, d_q(t) \frac{L_{\text{NWA},k,q}(x)}{\{k\}_q!}. \tag{406}$$

The following polynomials are influenced by Nörlund [102], who would maybe have denoted them by C instead of G.

Definition 69. The generating function for $G^{(n)}_{\text{NWA},\nu,q}(x)$ is

$$\frac{2^n}{(\text{E}_q(t\overline{2}_q) + 1)^n} \text{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu G^{(n)}_{\text{NWA},\nu,q}(x)}{\{\nu\}_q!}, \ |t| < \frac{\pi}{2}. \tag{407}$$

This implies

$$\nabla_{\text{NWA},2,q} G^{(n)}_{\text{NWA},\nu,q}(x) = G^{(n-1)}_{\text{NWA},\nu,q}(x). \tag{408}$$

$$\nabla_{\text{NWA},2,q} G_{\text{NWA},\nu,q}(x) = x^\nu. \tag{409}$$

This can be generalized to

Definition 70. The generating function for the $q-G$ polynomials of degree ν and order n $G^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is the following formula reminding of [102, p. 143 (78)].

$$\begin{gathered}\frac{2^n\mathrm{E}_q(xt)}{\prod_{k=1}^n(\mathrm{E}_q(\omega_k\overline{2}_q)+1)}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}G^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n),\\ |t|<\min(|\frac{\pi}{2\omega_1}|,\ldots,|\frac{\pi}{2\omega_n}|).\end{gathered}\tag{410}$$

Obviously, $G^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$G^{(1)}_{\mathrm{NWA},\nu,q}(x|\omega)=\omega^\nu G_{\mathrm{NWA},\nu,q}(\frac{x}{\omega}).\tag{411}$$

From

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\mathrm{NWA},2,q}}\ G^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=x^\nu\tag{412}$$

we obtain

$$\underset{\omega_1,\ldots,\omega_p}{\nabla^p_{\mathrm{NWA},2,q}}\ G^{(n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=G^{(n-p)}_{\mathrm{NWA},\nu,q}(x|\omega_{p+1},\ldots,\omega_n)\tag{413}$$

The following symbolic relations obtain.

Theorem 4.18. *The first formula is a q-analogue of* [101, p. 138] *($n=1$),* [102, p. 125]. *The second formula is a q-analogue of* [101, p. 149 (18)] *($n=1$),* [102, p. 124 (22)].

$$\frac{1}{2}\left[(G^{(n)}_{\mathrm{NWA},q}\oplus_q x\oplus_q\overline{2}_q)^\nu+(G^{(n)}_{\mathrm{NWA},q}\oplus_q x)^\nu\right]\ddot{=}(G^{(n-1)}_{\mathrm{NWA},q}\oplus_q x)^\nu.\tag{414}$$

$$\frac{1}{2}\left[(G^{(n)}_{\mathrm{NWA},q}\oplus_q\overline{2}_q)^\nu+G^{(n)}_{\mathrm{NWA},\nu,q}\right]\ddot{=}G^{(n-1)}_{\mathrm{NWA},\nu,q}.\tag{415}$$

Theorem 4.19. *A q-analogue of* [101, p. 137] *($n=1$),* [102, p. 124].

$$\begin{gathered}\nabla_{\mathrm{NWA},2,q}f(G^{(n)}_{\mathrm{NWA},q}(x))\equiv\\ \frac{1}{2}\left[f(G^{(n)}_{\mathrm{NWA},q}(x)\oplus_q\overline{2}_q)+f(G^{(n)}_{\mathrm{NWA},q}(x))\right]\ddot{=}f(G^{(n-1)}_{\mathrm{NWA},q}(x)).\end{gathered}\tag{416}$$

Theorem 4.20. *The following recurrence obtains, a q-analogue of* [101, p. 136, (20)], [102, p. 27].

$$\left[(G_{\mathrm{NWA},q}\oplus_q\overline{2}_q)^k+G_{\mathrm{NWA},k,q}\right]\ddot{=}2\delta_{0,k}.\tag{417}$$

Theorem 4.21.

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}G^{(n)}_{\mathrm{NWA},\nu,q}(x\oplus_q\overline{2k}_q)=x^\nu.\tag{418}$$

Proof. Use equations (408) and (371). □

Theorem 4.22. *Compare* [28, 3.15 p. 51], *where the corresponding formula for Euler polynomials was given.*

$$G_{\text{NWA},\nu,q}(x) \equiv \frac{2}{\mathrm{E}_q(\overline{2}_q D_q) + I} x^\nu = \frac{2}{E(\oplus_q)^{\overline{2}_q} + I} x^\nu \ddot{=} (x \oplus_q G_{\text{NWA},q})^\nu. \tag{419}$$

The following table lists some of the first $G_{\text{NWA},n,q}$. A q-analogue of the integers in [102, p. 27].

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
1	-1	$2^{-1}(-1+q)$	$q(1+q)$	$-2^{-2}(q^3-1)(1+q)^3$

We have the following operational representation.

Theorem 4.23.

$$G^{(n)}_{\text{NWA},\nu,q}(\omega_1, \ldots, \omega_n) \ddot{=} (\oplus_{q,l=1}^{n} \omega_l G_{\text{NWA},l,q})^\nu. \tag{420}$$

Corollary 4.24.

$$\mathrm{E}_q(tG_{\text{NWA},q}) \ddot{=} \frac{2}{\mathrm{E}_q(t\overline{2}_q) + 1}. \tag{421}$$

We will now give a few other q-analogues of equations for G polynomials. We start with

Definition 71. A q-analogue of [70, p. 91].

$$\tau_{\text{NWA},m,q}(n) \equiv \sum_{k=0}^{n-1} (-1)^k (\overline{2k+1_q})^m. \tag{422}$$

Theorem 4.25. *A q-analogue of* [89, p. 237], [98, p. 401].

$$\tau_{\text{NWA},m,q}(n) = \frac{(-1)^{n-1} G_{\text{NWA},m+1,q}(\overline{(2n+1)}_q) + G_{\text{NWA},m+1,q}(\overline{(1)}_q)}{2}. \tag{423}$$

Proof.

$$\begin{aligned} LHS &= \sum_{k=0}^{n-1} (-1)^k \nabla_{\text{NWA},2,q} G_{\text{NWA},m,q}((\overline{2k+1_q})) = \\ &\sum_{k=0}^{n-1} \frac{(-1)^k}{2} \left(G_{\text{NWA},m,q}(\overline{2k+1})_q + G_{\text{NWA},m,q}(\overline{2k+3_q}) \right) = RHS. \end{aligned} \tag{424}$$

□

Theorem 4.26. *The inverse NWA-difference is given by*

$$\triangle_{\mathrm{NWA},2,q}^{-1} f(\overline{2x+1_q})|_{x=0}^{n} = \sum_{k=0}^{n-1} f(\overline{2k+1_q}). \tag{425}$$

The inverse $\nabla_{\mathrm{NWA},2,q}$ *is given by*

$$\nabla_{\mathrm{NWA},2,q}^{-1}\left(\frac{f(\overline{1_q})+(-1)^{n-1}f(\overline{2n+1_q})}{2}\right) = \sum_{k=0}^{n-1}(-1)^k f(\overline{2k+1_q}). \tag{426}$$

Theorem 4.27. *Another integration by parts formula.*

$$\sum_{k=0}^{n-1} f(\overline{2k+1_q})\triangle_{\mathrm{NWA},q} g(\overline{2k+1_q})$$

$$= [f(\overline{2x+1_q})g(\overline{2x+1_q})]_0^n - \sum_{k=0}^{n-1} E(\oplus_q)^{\overline{2}_q} g(\overline{2k+1_q})\triangle_{\mathrm{NWA},2,q} f(\overline{2k+1_q}). \tag{427}$$

Theorem 4.28. *Compare* [53, p. 136] *($q=1$).*

$$\sum_{k=0}^{n-1}(-1)^k f(\overline{2k+1_q}) = \frac{(-1)^{x-1}}{2} f(G_{\mathrm{NWA},q}(\overline{2x+1_q}))|_0^n. \tag{428}$$

Proof. Apply $\nabla_{\mathrm{NWA},2,q}$ to both sides to get by (426)

$$\frac{f(\overline{0_q})+(-1)^{n-1}f(\overline{n_q})}{2} = \nabla_{\mathrm{NWA},q}\frac{(-1)^{x-1}}{2} f(G_{\mathrm{NWA},q}(\overline{2x+1_q}))|_0^n. \tag{429}$$

Finally use (408) with $n=1$. □

Example 6. Put $f(x)=x^m$ in (428) to get (423).

By (383) and (408) we have

$$\triangle_{\mathrm{NWA},2,q}^{n} L_{\mathrm{NWA},\nu,q}^{(n)}(x) = \frac{\{\nu\}_q!}{\{\nu-n\}_q!}x^{\nu-n},$$

$$\nabla_{\mathrm{NWA},2,q}^{n} G_{\mathrm{NWA},\nu,q}^{(n)}(x) = x^{\nu},$$

and we have

Definition 72. A q-analogue of the Lucas polynomial of negative order $-n$ is given by

$$L_{\mathrm{NWA},\nu,q}^{(-n)}(x) \equiv \frac{\{\nu\}_q!}{\{\nu+n\}_q!}\triangle_{\mathrm{NWA},2,q}^{n} x^{\nu+n}, \tag{430}$$

and the q-G polynomial of negative order $-n$ is given by

$$G^{(-n)}_{\mathrm{NWA},\nu,q}(x) \equiv \nabla^n_{\mathrm{NWA},2,q} x^\nu, \tag{431}$$

where $\nu, n \in \mathbb{N}$. This defines q-Lucas- and q-G polynomials of negative order as iterated $\triangle_{\mathrm{NWA},q}$ and $\nabla_{\mathrm{NWA},q}$ operating on positive integer powers of x. Furthermore,

$$L^{(-n)}_{\mathrm{NWA},\nu,q} \equiv L^{(-n)}_{\mathrm{NWA},\nu,q}(0), \tag{432}$$

$$G^{(-n)}_{\mathrm{NWA},\nu,q} \equiv G^{(-n)}_{\mathrm{NWA},\nu,q}(0). \tag{433}$$

A calculation shows that formulas (383) and (408) hold for negative orders too, and we get

$$L^{(-n-p)}_{\mathrm{NWA},\nu,q}(x \oplus_q y) \doteq (L^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q L^{(-p)}_{\mathrm{NWA},q}(y))^\nu, \tag{434}$$

and the same for q-G polynomials.

A special case is the following

$$L^{(-n)}_{\mathrm{NWA},\nu,q}(x \oplus_q y) \doteq (L^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q y)^\nu, \tag{435}$$

and the same for q-G polynomials.

Theorem 4.29. *A recurrence formula for the q-Lucas numbers and a recurrence formula for the q-G numbers.*

If $n, p \in \mathbb{Z}$ then

$$L^{(n+p)}_{\mathrm{NWA},\nu,q} \doteq (L^{(n)}_{\mathrm{NWA},q} \oplus_q L^{(p)}_{\mathrm{NWA},q})^\nu, \tag{436}$$

$$G^{(n+p)}_{\mathrm{NWA},\nu,q} \doteq (G^{(n)}_{\mathrm{NWA},q} \oplus_q G^{(p)}_{\mathrm{NWA},q})^\nu. \tag{437}$$

Theorem 4.30.

$$(x \oplus_q y)^\nu \doteq (L^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q L^{(n)}_{\mathrm{NWA},q}(y))^\nu, \tag{438}$$

$$(x \oplus_q y)^\nu \doteq (G^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q G^{(n)}_{\mathrm{NWA},q}(y))^\nu. \tag{439}$$

Proof. Put $p = -n$ in (434). □

In particular for $y = 0$

$$x^\nu \doteq (L^{(-n)}_{\mathrm{NWA},q} \oplus_q L^{(n)}_{\mathrm{NWA},q}(x))^\nu, \tag{440}$$

$$x^\nu \doteq (G^{(-n)}_{\mathrm{NWA},q} \oplus_q G^{(n)}_{\mathrm{NWA},q}(x))^\nu. \tag{441}$$

These recurrence formulas express q-Lucas- and q-G polynomials of order n without mentioning polynomials of negative order.

These can also be expressed in the form

$$x^\nu = \sum_{s=0}^{\nu} \frac{L^{(-n)}_{\mathrm{NWA},s,q}}{\{s\}_q!} D^s_q L^{(n)}_{\mathrm{NWA},\nu,q}(x), \tag{442}$$

$$x^{\nu}=\sum_{s=0}^{\nu}\frac{G_{\mathrm{NWA},s,q}^{(-n)}}{\{s\}_{q}!}D_{q}^{s}G_{\mathrm{NWA},\nu,q}^{(n)}(x). \tag{443}$$

We conclude that the q-Lucas- and q-G polynomials satisfy linear q-difference equations with constant coefficients.

$$\sum_{s=0}^{\nu}\binom{\nu}{s}_{q}L_{\mathrm{NWA},s,q}^{(n)}L_{\mathrm{NWA},\nu-s,q}^{(-n)}=\delta_{\nu,0}. \tag{444}$$

$$\sum_{s=0}^{\nu}\binom{\nu}{s}_{q}G_{\mathrm{NWA},s,q}^{(n)}G_{\mathrm{NWA},\nu-s,q}^{(-n)}=\delta_{\nu,0}. \tag{445}$$

Proof. Put $x=y=0$ in (438) and (439). □

Theorem 4.31. *Assume that $f(x)$ is analytic with q-Taylor expansion*

$$f(x)=\sum_{\nu=0}^{\infty}D_{q}^{\nu}f(0)\frac{x^{\nu}}{\{\nu\}_{q}!}. \tag{446}$$

Then we can express powers of $\triangle_{\mathrm{NWA},q}$ and $\nabla_{\mathrm{NWA},q}$ operating on $f(x)$ as powers of D_q as follows. These series converge when the absolute value of x is small enough.

$$\triangle_{\mathrm{NWA},2,q}^{n}f(x)=\sum_{\nu=0}^{\infty}D_{q}^{\nu+n}f(0)\frac{L_{\mathrm{NWA},\nu,q}^{(-n)}(x)}{\{\nu\}_{q}!}, \tag{447}$$

$$\nabla_{\mathrm{NWA},2,q}^{n}f(x)=\sum_{\nu=0}^{\infty}D_{q}^{\nu}f(0)\frac{G_{\mathrm{NWA},\nu,q}^{(-n)}(x)}{\{\nu\}_{q}!}. \tag{448}$$

Proof. Use (383), (122) and (408), (122) respectively. □

Now put $f(x)=\mathrm{E}_q(xt)$ to get the generating function for $L_{\mathrm{NWA},\nu,q}^{(-n)}(x)$ and $G_{\mathrm{NWA},\nu,q}^{(-n)}(x)$.

$$(\mathrm{E}_q(\overline{2_q}t)-1)^{n}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu+n}}{\{\nu\}_{q}!}L_{\mathrm{NWA},\nu,q}^{(-n)}(x), \tag{449}$$

$$\frac{(\mathrm{E}_q(\overline{2_q}t)+1)^{n}}{2^{n}}\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_{q}!}G_{\mathrm{NWA},\nu,q}^{(-n)}(x). \tag{450}$$

The reason for the difference in appearance compared to the original for the following equation is that one of the function arguments is a Ward number.

Theorem 4.32.

$$L^{(n)}_{\mathrm{NWA},\nu,q}(x \oplus_q \overline{2n}_q) = \sum_{k=0}^{n} \binom{n}{k} \frac{\{\nu\}_q!}{\{\nu-k\}_q!} L^{(n-k)}_{\mathrm{NWA},\nu-k,q}(x). \tag{451}$$

Proof. Use (98) and (383). □

Theorem 4.33.

$$\frac{\{\nu\}_q!}{\{\nu-n\}_q!} x^{\nu-n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} L^{(n)}_{\mathrm{NWA},\nu,q}(x \oplus_q \overline{2k}_q). \tag{452}$$

Proof. Use equation (101). □

Theorem 4.34.

$$\frac{1}{2} \sum_{k=0}^{\infty} \frac{L^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \triangle^n_{\mathrm{NWA},2,q} D^k_q f(y) = D^n_q f(x \oplus_q y). \tag{453}$$

Proof. Replace $f(x)$ by $f(x \oplus_q y)$ in (386).

$$\frac{1}{2}\Big(f(L^{(n)}_{\mathrm{NWA},q}(x) \oplus_q \overline{2}_q \oplus_q y) - f(L^{(n)}_{\mathrm{NWA},q}(x) \oplus_q y)\Big) \doteq D_q f(L^{(n-1)}_{\mathrm{NWA},q}(x) \oplus_q y). \tag{454}$$

Use the umbral formula (47) to get

$$\frac{1}{2} \sum_{k=0}^{\infty} \frac{L^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \triangle_{\mathrm{NWA},2,q} D^k_q f(y) = \sum_{k=0}^{\infty} \frac{L^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} D^{k+1}_q f(y). \tag{455}$$

Apply the operator $\triangle^{n-1}_{\mathrm{NWA},2,q}$ with respect to y to both sides and use (447).

$$\begin{aligned} &\frac{1}{2} \sum_{k=0}^{\infty} \frac{L^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \triangle^n_{\mathrm{NWA},q} D^k_q f(y) = \\ &\sum_{k=0}^{\infty} \frac{L^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!} \sum_{l=0}^{\infty} D^{k+l+n}_q f(0) \frac{L^{(-n+1)}_{\mathrm{NWA},l,q}(y)}{\{l\}_q!}. \end{aligned} \tag{456}$$

Finally use (438) to rewrite the righthand side. □

Corollary 4.35. *Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation*

$$\frac{1}{2} \begin{matrix} \triangle^n_{\mathrm{NWA},2,q} \\ \omega_1, \ldots, \omega_n \end{matrix} f(x) = D^n_q \varphi(x) \tag{457}$$

is given by

$$f(x \oplus_q y) = \sum_{k=0}^{\nu} \frac{L^{(n)}_{\mathrm{NWA},k,q}(x|\omega_1, \ldots, \omega_n)}{\{k\}_q!} D^k_q \varphi(y). \tag{458}$$

Proof. The LHS of (458) can be written as $\varphi(L_{\text{NWA},q}^{(n)}(x|\omega_1,\ldots,\omega_n)\oplus_q y)$, because if we apply $\frac{1}{2}\triangle_{\text{NWA},2,q,x}^{n}$ to both sides we get

$$\begin{gathered}\frac{1}{2}\underset{\omega_1,\ldots,\omega_n}{\triangle_{\text{NWA},2,q}^{n}} f(x\oplus_q y)=D_{q,x}^{n}\varphi(x\oplus_q y)=\\ \frac{1}{2}\underset{\omega_1,\ldots,\omega_n}{\triangle_{\text{NWA},2,q}^{n}}\varphi(L_{\text{NWA},q}^{(n)}(x|\omega_1,\ldots,\omega_n)\oplus_q y).\end{gathered} \tag{459}$$

□

Theorem 4.36.

$$\sum_{k=0}^{\infty}\frac{G_{\text{NWA},k,q}^{(n)}(x)}{\{k\}_q!}\nabla_{\text{NWA},2,q}^{n}D_q^k f(y)=f(x\oplus_q y). \tag{460}$$

Proof. Replace $f(x)$ by $f(x\oplus_q y)$ in (416).

$$\frac{1}{2}\Big(f(G_{\text{NWA},q}^{(n)}(x)\oplus_q \overline{2}_q\oplus_q y)+f(G_{\text{NWA},q}^{(n)}(x)\oplus_q y)\Big)\ddot{=}f(G_{\text{NWA},q}^{(n-1)}(x)\oplus_q y). \tag{461}$$

Use the umbral formula (47) to get

$$\sum_{k=0}^{\infty}\frac{G_{\text{NWA},k,q}^{(n)}(x)}{\{k\}_q!}\nabla_{\text{NWA},2,q}D_q^k f(y)=\sum_{k=0}^{\infty}\frac{G_{\text{NWA},k,q}^{(n-1)}(x)}{\{k\}_q!}D_q^k f(y). \tag{462}$$

Apply the operator $\nabla_{\text{NWA},2,q}^{n-1}$ with respect to y to both sides and use (448).

$$\begin{gathered}\sum_{k=0}^{\infty}\frac{G_{\text{NWA},k,q}^{(n)}(x)}{\{k\}_q!}\nabla_{\text{NWA},2,q}^{n}D_q^k f(y)=\\ \sum_{k=0}^{\infty}\frac{G_{\text{NWA},k,q}^{(n-1)}(x)}{\{k\}_q!}\sum_{l=0}^{\infty}D_q^{k+l}f(0)\frac{G_{\text{NWA},l,q}^{(-n+1)}(y)}{\{l\}_q!}.\end{gathered} \tag{463}$$

Finally use (439) to rewrite the righthand side. □

Corollary 4.37. *Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation*

$$\underset{\omega_1,\ldots,\omega_n}{\nabla_{\text{NWA},2,q}^{n}} f(x)=\varphi(x) \tag{464}$$

is given by

$$f(x\oplus_q y)=\sum_{k=0}^{\nu}\frac{G_{\text{NWA},k,q}^{(n)}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!}D_q^k\varphi(y). \tag{465}$$

Proof. The LHS of (465) can be written as $\varphi(G^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n)\oplus_q y)$, because if we apply $\nabla^n_{\text{NWA},2,q,x}$ to both sides we get

$$\begin{aligned}&\nabla^n_{\substack{\text{NWA},2,q\\ \omega_1,\ldots,\omega_n}} f(x\oplus_q y)=\varphi(x\oplus_q y)=\\ &\nabla^n_{\substack{\text{NWA},2,q\\ \omega_1,\ldots,\omega_n}} \varphi(G^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n)\oplus_q y).\end{aligned} \tag{466}$$

□

There are a few formulas similar to the Leibniz theorem. We can express the NWA difference operator in terms of the mean value operator and vice versa.

Theorem 4.38.

$$\triangle^n_{\text{NWA},2,q}(fg)=2^n\sum_{i=0}^n(-1)^{n-i}\binom{n}{i}\nabla^i_{\text{NWA},2,q}f(\nabla^{n-i}_{\text{NWA},2,q}E(\oplus_q)^{\overline{2i}_q})g. \tag{467}$$

Proof. Same as [81, p. 98, (13)]. □

Theorem 4.39.

$$\nabla^n_{\text{NWA},2,q}(fg)=(-\frac{1}{2})^n\sum_{i=0}^n(-2)^{i}\binom{n}{i}\nabla^i_{\text{NWA},2,q}f(\triangle^{n-i}_{\text{NWA},2,q}E(\oplus_q)^{\overline{2i}_q})g. \tag{468}$$

Proof. Same as [81, p. 99, (2)]. □

Theorem 4.40.

$$\nabla^n_{\text{NWA},2,q}(fg)=\sum_{i=0}^n(\frac{1}{2})^{i}\binom{n}{i}\triangle^i_{\text{NWA},2,q}f(\nabla^{n-i}_{\text{NWA},2,q}E(\oplus_q)^{\overline{2i}_q})g. \tag{469}$$

Proof. Same as [81, p. 99, (3)]. □

We can express the JHC difference operator in terms of the mean value operator and vice versa.

Theorem 4.41.

$$\triangle^n_{\text{JHC},2,q}(fg)=2^n\sum_{i=0}^n(-1)^{n-i}\binom{n}{i}\nabla^i_{\text{JHC},2,q}f(\nabla^{n-i}_{\text{JHC},2,q}E(\boxplus_q)^{\tilde{2i}_q})g. \tag{470}$$

Proof. Same as [81, p. 98, (13)]. □

Theorem 4.42.

$$\nabla^{n}_{\mathrm{JHC},2,q}(fg) = (-\frac{1}{2})^{n}\sum_{i=0}^{n}(-2)^{i}\binom{n}{i}\nabla^{i}_{\mathrm{JHC},2,q}f(\triangle^{n-i}_{\mathrm{JHC},2,q}E(\boxplus_{q})^{\tilde{2}i_{q}})g. \tag{471}$$

Proof. Same as [81, p. 99, (2)]. □

Theorem 4.43.

$$\nabla^{n}_{\mathrm{JHC},2,q}(fg) = \sum_{i=0}^{n}(\frac{1}{2})^{i}\binom{n}{i}\triangle^{i}_{\mathrm{JHC},2,q}f(\nabla^{n-i}_{\mathrm{JHC},2,q}E(\boxplus_{q})^{\tilde{2}i_{q}})g. \tag{472}$$

Proof. Same as [81, p. 99, (3)]. □

5. The Hahn-Cigler-Carlitz-Johnson approach

This chapter is a partial continuation of papers by Hahn [66, p. 6 2.2], Cigler [25, p. 102-104], Carlitz [17] and Johnson [78, p. 217]. The last three papers use the same q-Stirling numbers. We start with some definitions.

Definition 73. A q-analogue of the polynomial from [28, p. 20]. Compare [25, p. 102].

$$(x)_{k,q} \equiv \prod_{m=0}^{k-1}(x-\{m\}_{q}). \tag{473}$$

The following notation of Cigler [26, p. 107] will be used

Definition 74.

$$E^{l}_{\mathrm{C},q}f(x) \equiv f(xq^{l}+\{l\}_{q}). \tag{474}$$

Definition 75. [25, p. 102], [26, p. 107] This is a special case of the Hahn operator [66, p. 6 2.2].

$$\triangle_{\mathrm{H},q}f(x) \equiv \frac{f(qx+1)-f(x)}{1+(q-1)x}. \tag{475}$$

Example 7. [25, p. 102], a q-analogue of [28, p. 20, 2.5]

$$\triangle_{\mathrm{H},q}(x)_{k,q} = \{k\}_{q}(x)_{k-1,q}. \tag{476}$$

Example 8. [26, p. 107]

$$\triangle_{\mathrm{H},q}E_{\mathrm{C},q} = qE_{\mathrm{C},q}\triangle_{\mathrm{H},q}. \tag{477}$$

Example 9. A q-analogue of [3, p. 237, (27)].

$$(\{k\}_{q})_{l,q} = \{l\}_{q}!\binom{k}{l}_{q}q^{\binom{l}{2}}. \tag{478}$$

Theorem 5.1. *A q-Taylor formula from* [25, p. 103]*. A q-analogue of* [16, p. 11], [28, p. 25].

$$f(x)=\sum_{k=0}^{\infty}\frac{(\triangle_{\mathrm{H},q}^{k}f)(0)}{\{k\}_q!}(x)_{k,q}. \tag{479}$$

$$q^{\binom{n}{2}}\triangle_{\mathrm{H},q}^{n}=\frac{1}{(1+(q-1)x)^n}\prod_{k=0}^{n-1}(E_{\mathrm{C},q}-q^k). \tag{480}$$

Theorem 5.2.

$$\triangle_{\mathrm{H},q}^{n}(x)_{k,q}=q^{-\binom{n}{2}}(1+(q-1)x)^{-n}\sum_{l=0}^{n}(-1)^{n-l}q^{\binom{n-l}{2}}\binom{n}{l}_q(xq^l+\{l\}_q)_{k,q}. \tag{481}$$

Proof. Induction on n. □

We will rewrite the above theorem in a slightly different way

Theorem 5.3.

$$\triangle_{\mathrm{H},q}^{n}f(x)=q^{-\binom{n}{2}}(1+(q-1)x)^{-n}\sum_{l=0}^{n}(-1)^{n-l}q^{\binom{n-l}{2}}\binom{n}{l}_q E_{\mathrm{C},q}^{l}f(x). \tag{482}$$

Proof. The set of all $(x)_{k,q}$ is a basis for the formal power series. □

This formula can be inverted.

Theorem 5.4.

$$E_{\mathrm{C},q}^{n}f(x)=\sum_{i=0}^{n}q^{\binom{i}{2}}\binom{n}{i}_q(1+(q-1)x)^i\triangle_{\mathrm{H},q}^{i}f(x). \tag{483}$$

Proof. Use the above inversion theorem with $f(n)=\binom{n}{2}$. □

Corollary 5.5. *Another q-analogue of* [81, p. 97, 10]*. Let $f(x)$ and $g(x)$ be formal power series. Then*

$$\triangle_{\mathrm{H},q}^{n}(fg)=q^{-\binom{n}{2}}\sum_{i=0}^{n}q^{\binom{i}{2}+\binom{n-i}{2}}\binom{n}{i}_q\triangle_{\mathrm{H},q}^{i}f(\triangle_{H,q}^{n-i}E_{\mathrm{C},q}^{i})g. \tag{484}$$

Proof. Same as [81, p. 96 f]. □

Example 10. A q-analogue of [28, p. 20, 2.5], [94, p. 42].

$$\triangle_{\mathrm{H},q}\langle x+1-k;q\rangle_k=q^{x+1-k}\langle x+2-k;q\rangle_{k-1}(1-q^k). \tag{485}$$

Example 11. A q-analogue of [94, p. 26, 6].

$$\triangle_{\mathrm{H},q}^{n}\binom{x}{k}_q=\binom{x}{k-n}_q q^{n(x+1)-\frac{n(2k+1-n)}{2}}. \tag{486}$$

Theorem 5.6. *The analogue of integration by parts is a q-analogue of* [81, p. 105], [28, p. 21], [94, p. 41].

$$\sum_{k=0}^{n} f(k)\triangle_{\mathrm{H},q}g(k)=[fg]_0^n-\sum_{k=0}^{n}E_{\mathrm{C},q}g(k)\triangle_{\mathrm{H},q}f(k). \tag{487}$$

Remark 13. Lesky [88, p.138] has found a multiplicative version of this formula.

Definition 76. The q-Stirling number of the first kind $s(n,k)_q$ and the q-Stirling number of the second kind $S(n,k)_q$ are defined by [29, p. 38 (3.13-14)], [25, p. 103] and [78, p. 217, 4.11].

$$(x)_{n,q}\equiv\sum_{k=0}^{n}s(n,k)_q x^k, \tag{488}$$

$$x^n\equiv\sum_{k=0}^{n}S(n,k)_q(x)_{k,q}. \tag{489}$$

Remark 14. We use the same conventions for Stirling numbers as Cigler [28, p. 34], Jordan [81, p. 142], Gould [58], Vein & Dale [139, p. 306] and Milne [92, p. 90]. Other definitions usually differ in sign, as in [16, p. 114], [61], [20], [55] where all Stirling numbers are positive.

Remark 15. Schwatt [119, ch. 5] denotes the $S(n,k)$ by $a_{n,k}$ whithout knowing that they are Stirling numbers.

The following recursions follow at once [25, p. 103]. The second one, a q-analogue of [64, p. 256] and [119, p. 81 (4)], also occurred in [136, p. 85 13.1], [78, p. 213, 3.6], [85, 7215, exc 29].

$$s(n+1,k)_q=s(n,k-1)_q-\{n\}_q\ s(n,k)_q. \tag{490}$$

$$S(n+1,k)_q=S(n,k-1)_q+\{k\}_q\ S(n,k)_q. \tag{491}$$

The orthogonality relation is the following q-analogue of [81, p. 182], [28, p. 35]

Theorem 5.7. *The two q-Stirling numbers viewed as matrices are inverses of each other.*

$$\sum_k S(m,k)_q s(k,n)_q=\delta_{m,n}. \tag{492}$$

The following table lists some of the first $s(n,k)_q$. Compare [28, p. 34], [81, p. 144] and [139, p. 306].

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	-1	1	0	0
$n=3$	0	$1+q$	$-(2+q)$	1	0
$n=4$	0	$-\{3\}_q!$	$3+4q+3q^2+q^3$	$-3-2q-q^2$	1

The following table lists some of the first $S(n,k)_q$. Compare [28, p. 35].

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
$n=1$	0	1	0	0	0
$n=2$	0	1	1	0	0
$n=3$	0	1	$2+q$	1	0
$n=4$	0	1	$3+3q+q^2$	$3+2q+q^2$	1

There are a number of simple rules to check the computation of the q-Stirling numbers of the first kind, as the following q-analogues of [81, p. 145 ff] show. Put $x=1$ in (488) to obtain

$$\sum_{k=1}^{n} s(n,k)_q = 0,\ n>1. \tag{493}$$

Put $x=-1$ in (488) to obtain

$$\sum_{k=1}^{n} |s(n,k)_q| = (-1)^{n-1}\prod_{m=0}^{n-1}(1+\{m\}_q),\ q>0. \tag{494}$$

The q-Stirling numbers of the first kind have a particurlarly simple expression.

Theorem 5.8.

$$s(n,k)_q = (-1)^{k-n} e_{n-k}(1,\{2\}_q,\ldots,\{n-1\}_q), \tag{495}$$

where e_k denotes the elementary symmetric polynomial.

Proof. Use (488). □

Corollary 5.9.

$$s(n,1)_q = (-1)^{n-1}\{n-1\}_q!. \tag{496}$$

Proof. Put $k = 1$ in (495). □

There is an exact formula for $S(n,k)_q$, which will be quite useful.

Theorem 5.10. [29, p. 41 (3.35)], [25, p. 104], [17, p. 990, 3.3], [136, p. 86 13.2] *($c = 1$). A q-analogue of* [81, p. 169, (3)], [28, p. 37, 2.31], [135, p. 495], [119, p. 83 (19)]

$$S(n,k)_q = (\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \{k-i\}_q^n. \tag{497}$$

There are a number of simple rules to check the computation of the q-Stirling numbers of the second kind. We start with some q-analogues of [81, p. 170. f].

Theorem 5.11.

$$(-1)^n = \sum_{k=0}^{n} S(n,k)_q (-1)^k \prod_{m=0}^{k-1} (1 + \{k\}_q). \tag{498}$$

Proof. Put $x = -1$ in (489). □

Theorem 5.12.

$$S(n+1,n)_q - S(n,n-1)_q = \{n\}_q. \tag{499}$$

Proof. Put $k = n$ in (491). □

There are two kinds of generating functions for $S(n,k)_q$. The first one is

Theorem 5.13. [29, p. 42 (3.36)], [85, 7.2.1.5, answer 29], *a q-analogue of* [81, p. 175 (2)], [61, p. 337, 7.47], [136, p. 64, 2.6] *and* [28, p. 37, 2.30].

$$\sum_{n=m}^{\infty} S(n,m)_q t^n = \frac{t^m}{\prod_{l=1}^{m}(1 - t\{l\}_q)}, \; |t| < \frac{1}{m}. \tag{500}$$

This can be expressed in two other ways. A q-analogue of [81, p. 193. (1)].

$$\sum_{n=m}^{\infty} S(n,m)_q z^{-n-1} = \frac{1}{(z)_{m+1,q}} \equiv (z)_{-(m+1),q}, \; z > m. \tag{501}$$

A q-analogue of [81, p. 193. (2)] and [28, p. 36, 2.29].

$$\sum_{n=m}^{\infty} S(n,m)_q(-x)^{-n} = \frac{(-1)^m}{\prod_{l=1}^{m}(x+\{l\}_q)},\ x > m. \tag{502}$$

By the orthogonality relation we obtain a q-analogue of [81, p. 193. (3)].

$$x^{-k} = \sum_{m=k}^{\infty} \frac{|s(m,k)_q|}{\prod_{l=1}^{m}(x+\{l\}_q)}. \tag{503}$$

The q-Stirling numbers can be used to obtain a number of exact formulas for q-derivatives and q-integrals as in

Theorem 5.14.

$$\int_0^1 (t)_{n,q}\, d_q(t) = \sum_{k=1}^{n} \frac{s(n,k)_q}{\{k+1\}_q}. \tag{504}$$

Proof. q-integrate (488). □

The formula (501) serves as definition of q-reciprocal factorial and we obtain

Theorem 5.15. *A q-analogue of* [81, p. 194 (5)].

$$D_q^s \frac{1}{(z)_{m+1,q}} = \sum_{n=m}^{\infty} S(n,m)_q \{-n-s\}_{s,q} z^{-n-1-s}. \tag{505}$$

Theorem 5.16. *A q-analogue of* [81, p. 194 (6)].

$$\int^z \frac{1}{(t)_{m+1,q}}\, d_q(t) = \sum_{n=m}^{\infty} \frac{S(n,m)_q}{\{-n\}_q z^n} + k. \tag{506}$$

The second generating functions for $S(n,m)_q$ is

Theorem 5.17. [29, p. 42 (3.38)]. *A q-analogue of* [31, p. 206 (2a)], [136, p. 64, 2.7].

$$\sum_{n=k}^{\infty} \frac{S(n,k)_q t^n}{\{n\}_q!} = (\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \mathrm{E}_q(t\{k-i\}_q). \tag{507}$$

Proof.

$$LHS = \sum_{n=k}^{\infty} \frac{t^n}{\{n\}_q!}(\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \{k-i\}_q^n = \\ (\{k\}_q! q^{\binom{k}{2}})^{-1} \sum_{i=0}^{k} \binom{k}{i}_q (-1)^i q^{\binom{i}{2}} \sum_{n=0}^{\infty} \frac{(t\{k-i\}_q)^n}{\{n\}_q!} = RHS. \tag{508}$$

□

There is a generating function for q-Stirling numbers of the first kind.

Theorem 5.18. *A q-analogue of* [81, p. 185 (1)].

$$\sum_{k=m}^{n} s(n,k)_q \binom{k}{m} q^m = q^{n-1} s(n-1,m)_q + q^n s(n-1,m-1)_q. \tag{509}$$

This formula can be inverted.

Theorem 5.19. *A q-analogue of* [81, p. 185].

$$s(n,k)_q = \sum_{m=k}^{n} q^{n-m-1}(-1)^{m+k} \binom{m}{k} [s(n-1,m)_q + qs(n-1,m-1)_q]. \tag{510}$$

Corollary 5.20. *A q-analogue of* [81, p. 186 (4)]

$$\sum_{k=1}^{n} s(n,k)_q k = \begin{cases} 1, & n=1; \\ q^{n-2}(-1)^n \{n-2\}_q! & n>1 \end{cases} \tag{511}$$

Proof. Put $m=1$ in (509). □

Corollary 5.21. *A q-analogue of* [81, p. 186 (5)]

$$\sum_{n=2}^{m} S(m,n)_q \; q^{n-2}(-1)^n \{n-2\}_q! = m-1. \tag{512}$$

Proof. Apply $\sum_{n=1}^{m} S(m,n)_q$ to both sides of (511). Then use the orthogonality relation. □

Theorem 5.22. *A q-analogue of* [81, p. 187 (10)]

$$\sum_{k=1}^{n} s(n,k)_q S(k+1,i)_q = \{n\}_q \binom{0}{n-i} + \binom{0}{n+1-i}. \tag{513}$$

Theorem 5.23. *A q-analogue of* [81, p. 188 (11)]

$$\sum_{k=0}^{n} S(n,k)_q [s(k+1,l)_q + \{k\}_q s(k,l)_q] = \delta_{l,n+1}. \tag{514}$$

Theorem 5.24. *A q-analogue of* [81, p. 188 (15)]

$$\sum_{k=1}^{n+1} S(n+1,k)_q = \sum_{k=1}^{n} (1+\{k\}_q) S(n,k)_q, n>0. \tag{515}$$

The following operator [77] will also be useful. In its earliest form with $q=1$ it dates back to Euler and Abel [1, B. 2, p. 41], who used it in differential equations.

Definition 77.

$$\theta_q \equiv x D_q. \tag{516}$$

Theorem 5.25. *A q-analogue of* [58, p. 455, 4.8], [127, p. 181], [136, p. 64, 2.1], [119, p. 81], [81, p. 196 (2)].

$$\theta_q^n = \sum_{k=0}^{n} S(n,k)_q \; q^{\binom{k}{2}} x^k D_q^k. \tag{517}$$

Proof. Induction. □

This leads to the following inverse formula.

Theorem 5.26. *Cigler* [29, p. 37 (3.9)]. *A q-analogue of* [132, p 548], [127, p 183].

$$q^{\binom{n}{2}} x^n D_q^n = \sum_{k=1}^{n} s(n,k)_q \theta_q^k. \tag{518}$$

Proof. Use the orthogonality relation for q-Stirling numbers. □

The previous formula can be expressed in another way.

Theorem 5.27. *Jackson* [76, p. 305]. *A q-analogue of the 1844 Boole formula* [15] [127, p 183], [24, p 24, (2.1)].

$$q^{\binom{n}{2}} x^n D_q^n = \prod_{k=0}^{n-1} (\theta_q - \{k\}_q). \tag{519}$$

Proof. Use (488). □

Example 12. A q-analogue of [81, p. 196]. Let $f(x) = (x \oplus_q 1)^n$ and apply (517) to get

$$\sum_{k=0}^{n} \binom{n}{k}_q \; x^k \{k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q \; q^{\binom{k}{2}} x^k \{n-k+1\}_{k,q} (x \oplus_q 1)^{n-k}. \tag{520}$$

Put $x = 1$ to get

$$\sum_{k=0}^{n} \binom{n}{k}_q \{k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q \, q^{\binom{k}{2}} \{n-k+1\}_{k,q} (1 \oplus_q 1)^{n-k}. \quad (521)$$

Put $x = -1$ to get

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k \{k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q \, q^{\binom{k}{2}} (-1)^k \{n-k+1\}_{k,q} (1 \ominus_q 1)^{n-k}. \quad (522)$$

Example 13. Let $f(x) = (x \boxplus_q 1)^n$ and apply (517) to get

$$\sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} x^{n-k} \{n-k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q \, q^{\binom{k}{2}} x^k \{n-k+1\}_{k,q} (x \boxplus_q 1)^{n-k}. \quad (523)$$

Put $x = 1$ to get

$$\sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} \{n-k\}_q^m = \sum_{k=0}^{\min(m,n)} S(m,k)_q \, q^{\binom{k}{2}} \{n-k+1\}_{k,q} (1 \boxplus_q 1)^{n-k}. \quad (524)$$

Put $x = -1$ to get (497).

We continue with a few equations with operator proofs in the spirit of Gould and Schwatt.

Theorem 5.28. *A q-analogue of* [58, p 455, 4.9].

$$\sum_{k=0}^{n} \{k\}_q{}^p x^k = \sum_{k=0}^{n} S(n,k)_q \, q^{\binom{k}{2}} x^k D_q^k \left(\frac{x^{n+1} - 1}{x - 1} \right). \quad (525)$$

We obtain the following limit

Theorem 5.29.

$$\sum_{k=0}^{n} \{k\}_q{}^p = \sum_{k=0}^{n} S(n,k)_q \, q^{\binom{k}{2}} \lim_{x \to 1} D_q^k \left(\frac{x^{n+1} - 1}{x - 1} \right). \quad (526)$$

Theorem 5.30. *A q-analogue of* [58, p. 456, 4.10], [119, p. 85], [59, p. 490], [54].

$$\sum_{k=0}^{\infty} \{k\}_q{}^p x^k = \sum_{k=0}^{p} S(p,k)_q q^{\binom{k}{2}} \frac{x^k \{k\}_q!}{(x;q)_{k+1}}, \ |x| < 1. \quad (527)$$

Proof. Let $n \to \infty$ in (525). ∎

Theorem 5.31. *A q-analogue of* [58, p. 456, 4.11].

$$\theta_q^n \mathrm{E}_q(x) = \mathrm{E}_q(x) \sum_{k=0}^{n} S(n,k)_q \; q^{\binom{k}{2}} x^k. \tag{528}$$

The following q-analogue of Bell numbers is the same as Milne [92, p. 99].

Definition 78. Compare [25, p. 104] ($x = 1$). The q-Bell number is given by

$$B_q(n) \equiv \sum_{k=0}^{n} S(n,k)_q \; q^{\binom{k}{2}}. \tag{529}$$

Theorem 5.32. *The q-Dobinsky theorem* [92, p. 108, 4.5] *is q-analogue of* [119, p. 84]

$$B_q(n) = \mathrm{E}_{\frac{1}{q}}(-1) \sum_{k=0}^{\infty} \frac{\{k\}_q^n}{\{k\}_q!}. \tag{530}$$

This can be generalized to

Theorem 5.33. *A q-analogue of* [119, p 84].

$$\sum_{k=0}^{\infty} \frac{{\{k\}_q}^p x^k}{\{k\}_q!} = \sum_{k=0}^{p} S(p,k)_q q^{\binom{k}{2}} x^k \mathrm{E}_q(x)(x;q)_{k+1}. \tag{531}$$

We will now define another q-analogue of sums of powers than in chapter 2. It is a q-analogue of [28, p. 13], [138, p. 575], [119, p. 86]. The aim is to find equations similar to the $q = 1$ case. This function was also treated by Kim [83] from a different point of view.

Definition 79. Carlitz [17, p. 994-995]

$$S_{\mathrm{C},m,q}(n) \equiv \sum_{i=0}^{n-1} \{i\}_q^m q^i, \;\; S_{\mathrm{C},0,q}(1) \equiv 1. \tag{532}$$

Lemma 5.34. *A q-analogue of* [28, p. 20, 2.6]

$$\sum_{i=0}^{n-1} (\{i\}_q)_{j,q} q^i = \frac{(\{n\}_q)_{j+1,q}}{\{j+1\}_q}, \;\; j < n. \tag{533}$$

Proof. Use induction on n. □

The Carlitz sum can be expressed as a double sum of q-Stirling numbers.

Corollary 5.35. *A q-analogue of* [28, p. 35]. *Compare* [17, p. 994, 6.1].

$$S_{\mathrm{C},m,q}(n)=\sum_{j}\frac{S(m,j)_q}{\{j+1\}_q}\sum_{l=0}^{j+1}s(j+1,l)_q\{n\}_q^l. \tag{534}$$

Proof.

$$\begin{aligned}LHS&=\sum_{i=1}^{n-1}\sum_{j=0}^{m}S(m,j)_q(\{i\}_q)_{j,q}q^i\\&=\sum_{j=0}^{m}S(m,j)_q\frac{(\{n\}_q)_{j+1,q}}{\{j+1\}_q}=RHS.\end{aligned} \tag{535}$$

□

Corollary 5.36. *A q-analogue of* [82, p. 98].

$$S_{\mathrm{C},2,q}(n)=\frac{(\{n\}_q)_{3,q}}{\{3\}_q}+\frac{(\{n\}_q)_{2,q}}{\{2\}_q},\ n\geq 2. \tag{536}$$

Proof. By (533),

$$\sum_{i=0}^{n-1}\{i\}_q(\{i\}_q-1)q^i=\frac{(\{n\}_q)_{3,q}}{\{3\}_q} \tag{537}$$

□

Remark 16. In general, $S_{\mathrm{C},m,q}(n)$ contains the factor $(\{n\}_q)_{2,q}$.

6. The SCGMZP approach

In this Chapter, we mainly consider functions $f,g\in\mathbb{C}[[q^x]]$. We use the following abbreviation
Schendel–Carlitz–Gould–Milne–Zeng–Phillips=SCGMZP, as in the title.

The following operators were introduced by Carlitz [17, p. 988] 1948. Schendel [116], Gould [54], Milne [92], Zeng [142] and Phillips [104] used the same technique. Applications from approximation theory can be found in Phillips [104]. Observe that the q-Stirling number of the second kind used by Milne [92, p. 93] is $q^{\binom{k}{2}}S(n,k)_q$.

Definition 80. The Carlitz–Gould q-difference is defined by

$$\triangle_{\mathrm{CG},q}f(x)\equiv f(x+1)-f(x),\ \triangle_{\mathrm{CG},q}^{n+1}f(x)\equiv\triangle_{\mathrm{CG},q}^{n}f(x+1)-q^n\triangle_{\mathrm{CG},q}^{n}f(x). \tag{538}$$

Remark 17. We get the above definition by putting $y=-1$ in Schendel [116].

Theorem 6.1. *The following q-Taylor formula applies* [17, 2.5 p. 988], [56, 7.2, p. 856], [54, 2.11, p. 91].

$$f(x+y) = \sum_{k=0}^{\infty} \frac{\triangle_{\text{CG},q}^{k} f(y)}{\{k\}_q!} \{x-k+1\}_{k,q}. \tag{539}$$

Theorem 6.2. [142], [54, 2.10, p. 91], [104, p. 46 1.118], [92, p. 91] *and a q-analogue of* [28, p. 26]. *Compare* [116, p. 82]

$$\triangle_{\text{CG},q}^{n} f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} E^{n-k} f(x), \tag{540}$$

where the shift operator E is given by

$$E^n f(x) \equiv f(x+n). \tag{541}$$

Proof. Use induction. □

This formula can be inverted.

Theorem 6.3.

$$E^n f(x) = \sum_{i=0}^{n} \binom{n}{i}_q \triangle_{\text{CG},q}^{i} f(x). \tag{542}$$

Proof. This is the general inversion formula again, compare the corrected version of [52, p. 244]. □

Corollary 6.4. [104, p. 47 1.122], *a q-analogue of* [81, p. 97, 10], [28, p. 27, 2.13], [94, p. 35, 2]. *Assume that the functions $f(x)$ and $g(x)$ depend on q^x. Then*

$$\triangle_{\text{CG},q}^{n}(fg) = \sum_{i=0}^{n} \binom{n}{i}_q \triangle_{\text{CG},q}^{i} f \triangle_{\text{CG},q}^{n-i} E^i g. \tag{543}$$

Proof. Compare [81, p. 96 f].

$$\begin{aligned} LHS = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} E^{n-k} f \; E^{n-k} g = \\ \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} \sum_{i=0}^{n-k} \binom{n-k}{i}_q \triangle_{\text{CG},q}^{i} f \; E^{n-k} g = \\ \sum_{i=0}^{n} \binom{n}{i}_q \triangle_{\text{CG},q}^{i} f \; \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k}_q q^{\binom{k}{2}} E^{n-k} g = RHS. \end{aligned} \tag{544}$$

□

Example 14.

$$\triangle_{\mathrm{CG},q}^{n}(q^{mx}) = \begin{cases} q^{mx}(-1)^n q^{\binom{n}{2}}\langle 1+m-n;q\rangle_n, \ m \geq n. \\ 0, \ m < n. \end{cases} \tag{545}$$

Proof. Use (540). □

Theorem 6.5. [92, p. 93].

$$S(n,k)_q = (\{k\}_q! q^{\binom{k}{2}})^{-1} \triangle_{\mathrm{CG},q}^{n}\{x\}_q^n | x=0. \tag{546}$$

We will now follow [119, ch. 5] and develop a calculus for the Carlitz function $S_{\mathrm{C},m,q}(n)$ from the previous chapter. First a lemma.

Lemma 6.6. *A q-analogue of* [119, p. 86, (50)].

$$\sum_{s=k}^{n-1} \binom{s}{k}_q q^s = \binom{n}{k+1}_q q^k. \tag{547}$$

Theorem 6.7. *A q-analogue of* [119, p. 86, (51)].

$$S_{\mathrm{C},m,q}(n) = \sum_{k=0}^{m} (-1)^k q^k \binom{n}{k+1}_q \sum_{a=0}^{k} (-1)^a \binom{k}{a}_q q^{\binom{k-a}{2}} \{a\}_q^m. \tag{548}$$

Proof. Write the LHS as

$$\theta_q^m \sum_{k=1}^{n-1} (xq)^k |_{x=1}, \tag{549}$$

and use (517), (547) and (497). □

Theorem 6.8. *A q-analogue of* [119, p. 87 (63)].

$$S_{\mathrm{C},m,q}(n) = \sum_{k=0}^{n} \binom{n}{k}_q \sum_{a=0}^{k} (-1)^a \binom{k}{a}_q q^{\binom{a}{2}} \sum_{i=1}^{k-a-1} \{i\}_q^m q^i. \tag{550}$$

Proof. Use (539) and (540). □

Now let $b_{m,k,q}$ denote the coefficient of $q^k \binom{n}{k+1}_q$ in $S_{\mathrm{C},m,q}(n)$. Then by (548) the following recurrence obtains, which is almost a q-analogue of [119, p. 88 (69)].

$$b_{m,k,q} - \{k\}_q b_{m-1,k,q} = q^{k-1}\{k\}_q b_{m-1,k-1,q}. \tag{551}$$

We obtain the following expressions for $S_{\mathrm{C},m,q}(n)$ expressed as sums of q-binomial coefficients.

Theorem 6.9. *Almost a q-analogue of* [119, p. 88 (69)].

$$S_{\mathrm{C},1,q}(n) = q\binom{n}{2}_q. \tag{552}$$

$$S_{\mathrm{C},2,q}(n) = q\binom{n}{2}_q + q^3(1+q)\binom{n}{3}_q. \tag{553}$$

$$\begin{aligned} S_{\mathrm{C},3,q}(n) = q\binom{n}{2}_q + q^3(1+q+(1+q)^2)\binom{n}{3}_q + \\ q^6(1+q)(1+q+q^2)\binom{n}{4}_q. \end{aligned} \tag{554}$$

$$\begin{aligned} S_{\mathrm{C},4,q}(n) = q\binom{n}{2}_q + q^3(1+q)(2+q+(1+q)^2)\binom{n}{3}_q + q^6(1+q+q^2) \\ \times (1+q+(1+q)^2+\{3\}_q!)\binom{n}{4}_q + q^{10}\{4\}_q!\binom{n}{5}_q. \end{aligned} \tag{555}$$

By the q-Pascal identity we obtain the following q-analogue of [95, p. 14].

Theorem 6.10.

$$S_{\mathrm{C},2,q}(n) = q^3\binom{n}{3}_q + q\binom{n+1}{3}_q. \tag{556}$$

$$S_{\mathrm{C},3,q}(n) = q^6\binom{n}{4}_q + 2q^3(1+q)\binom{n+1}{4}_q + q\binom{n+2}{4}_q. \tag{557}$$

$$\begin{aligned} S_{\mathrm{C},4,q}(n) = q^{10}\binom{n}{5}_q + q^6(3+5q+3q^2)\binom{n+1}{5}_q + \\ + q^3(3+5q+3q^2)\binom{n+2}{5}_q + q\binom{n+3}{5}_q. \end{aligned} \tag{558}$$

We can now introduce the sum operator mentioned in the introduction.

Definition 81. The inverse CG difference is defined by

$$\triangle_{\mathrm{CG},q}^{-1} f(k)|_0^n \equiv \sum_0^n f(x)\delta_q(x) \equiv \sum_{k=0}^{n-1} f(k). \tag{559}$$

Example 15. Compare [92, p. 92, 1.11].

$$\triangle_{\mathrm{CG},q,x}^{m}\langle x+\gamma;q\rangle_n=\langle n-m+1;q\rangle_m\langle x+m+\gamma;q\rangle_{n-m}q^{m(x+\gamma+m-1)},\ m\leq n. \tag{560}$$

This is equivalent to the following formula from [54, 2.12, p. 91], a q-analogue of [94, p. 26, 6].

$$\triangle_{\mathrm{CG},q}^{m}\binom{x}{n}_q=\binom{x}{n-m}_q q^{m(x+m-n)},\ m\leq n. \tag{561}$$

Proof. Use induction. □

Example 16.

$$\triangle_{\mathrm{CG},q}^{-1}(1-q^l)\langle n+1;q\rangle_{l-1}\ q^n\equiv\sum_{k=0}^{n-1}(1-q^l)\langle k+1;q\rangle_{l-1}q^k=\langle n;q\rangle_l. \tag{562}$$

Corollary 6.11.

$$\{n\}_q\{n+1\}_q=\{2\}_q\sum_{i=1}^{n}\{i\}_q q^{i-1}. \tag{563}$$

$$\sum_{i=1}^{n}\{i\}_q q^{2i}=\frac{\{n\}_{2,q}}{\{2\}_q}-\frac{\{n\}_{3,q}(1-q)}{\{3\}_q}. \tag{564}$$

It is possible to develop a calculus similar to $S_{\mathrm{C},m,q}(n)$ for the sum (564), but we have not pursued this path.

By (540) and (560) we obtain the following

Theorem 6.12. [99, p. 110]

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}\langle x+1;q\rangle_{m-n}\langle x-y+1-n+m;q\rangle_n= \langle y-m+1;q\rangle_m q^{m(x-y+m)},\ x,y\in\mathbb{C}. \tag{565}$$

7. The Jackson q-derivative as difference operator

This chapter will be about how the Jackson q-derivative can be used as difference operator operating on the space of all q-shifted factorials. We illustrate the technique with some examples. The similarity with the operator from the previous chapter is striking and will apparently lead to many multiple q-equations. However it turns out that most of these are doublets, as is shown in the example from the last chapter.

For functions of q^x, the Cigler operator ϵ [25] will be replaced by E in q-Leibniz theorems as below.

Theorem 7.1.

$$D^n_{q,q^x}\langle\gamma+x;q\rangle_k=(-1)^n\{k-n+1\}_{n,q}\langle\gamma+x+n;q\rangle_{k-n}q^{\binom{n}{2}+n\gamma}. \quad (566)$$

Example 17. We apply the operator D^m_{q,q^x} to (565). Then

$$LHS=\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}\sum_{i=0}^{m}\binom{m}{i}_q D^i_{q,q^x}\langle x+1;q\rangle_{m-n}$$
$$E^iD^{m-i}_{q,q^x}\langle x-y+1-n+m;q\rangle_n=\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}\times$$
$$\binom{m}{m-n}_q(-1)^m\{m-n\}_q!q^{\binom{m-n}{2}+m-n}\{n\}_q!\;q^{\binom{n}{2}+n(-y+1-n+m)} \quad (567)$$
$$=(-1)^mq^{\binom{m}{2}+m}\{m\}_q!\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}-ny}=$$
$$q^{m^2-my}\langle y+1-m;q\rangle_m\{m\}_q!=RHS.$$

Now instead rewrite (565) in the form

$$\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}\langle y-x-m;q\rangle_nq^{n(x+m)+y(m-n)}=$$
$$\langle y-m+1;q\rangle_mq^{m(x+m)},\;x,y\in\mathbb{C}, \quad (568)$$

and operate with D^m_{q,q^y} on both sides to obtain

$$LHS=\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}q^{n(x+m)}\sum_{i=0}^{m}\binom{m}{i}_q D^i_{q,q^y}q^{y(m-n)}$$
$$E^iD^{m-i}_{q,q^y}\langle y-x-m;q\rangle_n=\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}q^{n(x+m)}\times$$
$$\binom{m}{m-n}_q\{m-n\}_q!(-1)^n\{n\}_q!q^{\binom{n}{2}-n(x+m)} \quad (569)$$
$$=\{m\}_q!\sum_{n=0}^{m}\binom{m}{n}_q\langle x+1;q\rangle_{m-n}(-1)^nq^{\binom{n}{2}}.$$

The RHS is

$$(-1)^m\{m\}_q!q^{\binom{m}{2}+m(x+1)}. \quad (570)$$

After simplification this last equality is equivalent to a confluent form of the second q-Vandermonde identity.

Inspired by the previous calculation we make the following

Definition 82. The Jackson q-difference is defined by

$$\begin{aligned}&\triangle_{\mathrm{J},x,q} f(q^x) \equiv \triangle_{\mathrm{J},q} f(q^x)\\ &\equiv (f(q^{x+1}) - f(q^x))q^{-x} \equiv -(1-q)D_{q,q^x} f(q^x),\end{aligned} \tag{571}$$

$$\triangle_{\mathrm{J},q}^{n+1} = \triangle_{\mathrm{J},q}\triangle_{\mathrm{J},q}^{n}. \tag{572}$$

The following equation obtains.

Theorem 7.2.

$$\triangle_{\mathrm{J},q}\left(q^{\binom{k}{2}}\binom{x}{k}_q\right) = q^{\binom{k-1}{2}}\binom{x}{k-1}_q. \tag{573}$$

Proof. Use the q-Pascal identity. □

Corollary 7.3.

$$\triangle_{\mathrm{J},q}^{m}\binom{x}{n}_q = \binom{x}{n-m}_q q^{-mn+\binom{m+1}{2}},\ m \le n. \tag{574}$$

Theorem 7.4.

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}}\binom{x}{n}_q t^n = \mathrm{e}_q(-tq^x \boxplus_q t) \tag{575}$$

Proof. Use the q-binomial theorem. □

Theorem 7.5. *The following q-Taylor formula applies.*

$$f(x) = \sum_{k=0}^{\infty}\binom{x}{k}_q q^{\binom{k}{2}}\triangle_{\mathrm{J},q}^{k} f(0). \tag{576}$$

Theorem 7.6. *A q-analogue of* [28, p. 26]. *Compare* [116, p. 82]

$$\triangle_{\mathrm{J},q}^{n} f(q^x) = q^{-nx-\binom{n}{2}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\binom{k}{2}}E^{n-k}f(q^x). \tag{577}$$

Proof. Use the corresponding equation for the q-derivative. □

This formula can be inverted.

Theorem 7.7.

$$E^n f(q^x) = \sum_{k=0}^{n} q^{xk+\binom{k}{2}}\binom{n}{k}_q \triangle_{\mathrm{J},q}^{k} f(q^x). \tag{578}$$

Corollary 7.8. *A q-analogue of* [81, p. 97, 10], [28, p. 27, 2.13], [94, p. 35, 2].

$$\triangle_{\mathrm{J},q}^{n}(f(q^x)g(q^x)) = \sum_{i=0}^{n}\binom{n}{i}_q \triangle_{\mathrm{J},q}^{i} f(q^x)(\triangle_{\mathrm{J},q}^{n-i}E^i)g(q^x). \tag{579}$$

Proof. Use the Leibniz theorem for the q-derivative. □

Example 18.

$$\triangle_{\mathrm{J},q}^{n}(q^{mx}) = \begin{cases} q^{x(m-n)}(-1)^n\langle 1+m-n;q\rangle_n, \ m \geq n. \\ 0, \ m < n. \end{cases} \tag{580}$$

8. Applications

The technique developed leads to easy proofs of q-binomial coefficient identities. The following example can also be proved from the q-Vandermonde identity.

Example 19. A q-analogue of the important formula [28, p. 27], [109, p. 15, (9)], [112, p. 65].

$$\begin{aligned} \binom{x}{m}_q \binom{x}{n}_q = \sum_{k=0}^{m+n} \mathrm{QE}\left((k-n)(k-m)\right) \times \\ \binom{k}{n}_q \binom{n}{m+n-k}_q \binom{x}{k}_q. \end{aligned} \tag{581}$$

Proof. We have

$$\begin{aligned} \triangle_{\mathrm{CG},q}^{k} \binom{x}{m}_q \binom{x}{n}_q = \sum_{l=0}^{k} \binom{k}{l}_q \mathrm{QE}\left(l(x+l-n)+(k-l)(x+k-m)\right) \times \\ \binom{x+l}{m-k+l}_q \binom{x}{n-l}_q. \end{aligned} \tag{582}$$

Now use the q-Taylor formula (539) with $y=0$, $f(x) = \binom{x}{m}_q \binom{x}{n}_q$. □

Remark 18. If we use $\triangle_{\mathrm{J},q}$ instead, we get

$$\begin{aligned} \binom{x}{m}_q \binom{x}{n}_q = \sum_{k=0}^{m+n} \mathrm{QE}\left(-\binom{m}{2} - \binom{n}{2} + \binom{m+n-k}{2} + \binom{k}{2}\right) \times \\ \binom{k}{n}_q \binom{n}{m+n-k}_q \binom{x}{k}_q. \end{aligned} \tag{583}$$

This equation is however equivalent to (581).

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